The étale fundamental group, étale homotopy and anabelian geometry

Master's thesis in Mathematics

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Abstract

In 1983 Grothendieck wrote a letter to Faltings, [Gro83], outlining what is today known as the anabelian conjectures. These conjectures concern the possibility to reconstruct curves and schemes from their étale fundamental group. Although Faltings never replied to the letter, his student Mochizuki began working on it. A major achievement by Mochizuki and Tamagawa was to prove several important versions of these conjectures.

In this thesis we will first introduce Grothendieck's Galois theory with the aim to define the étale fundamental group and formulate Mochizuki's result. After recalling some necessary homotopy theory, we will introduce the étale homotopy type, which is an extension of the étale fundamental group developed by Artin, Mazur and Friedlander. This is done in order to describe some recent work by Schmidt and Stix that improves on the results of Mochizuki and Tamagawa by extending them from étale fundamental groups to étale homotopy types of certain (possibly higher-dimensional) schemes.

Sammanfattning

Den étala fundamentalgruppen, étalehomotopi och anabelsk geometri

I ett brev till Faltings 1983, [Gro83], lade Grothendieck grunden till det som idag kallas anabelsk geometri. I brevet presenterar han ett antal förmodningar som handlar om möjligheten att återskapa kurvor och scheman från deras étala fundamentalgrupper. Faltings svarade aldrig på brevet, men hans student Mochizuki bevisade ett antal viktiga specialfall.

I den här uppsatsen introducerar vi först Grothendiecks Galoisteori för att definera den étala fundamentalgruppen och formulera Mochizukis resultat. Sedan går vi igenom grundläggande homotopiteori, som behövs för att introducera étalehomotopi, som utvecklats av Artin, Mazur och Friedlander. Med dessa hjälpmedel tittar vi närmare på ett resultat av Stix och Schmidt som bygger på Mochizukis resultat och utvidgar det från fundamentalgrupper till homotopityper.

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Contents

Chapter 1

Introduction

1.1. Prelude

The study of fundamental groups was an early and important part of algebraic topology introduced by Poincaré. As topological and continuous problems usually are very hard, approaching topological problems with the power of algebra can reduce hard problems into simple algebra.

A beautiful example of this process is the two-dimensional Brouwer fixed point theorem, which is a statement about *all* possible continuous maps $f : D \to D$ from the closed Euclidean disk D to itself. A simple fact of such a map is that if it does not have any fixed points, we could use it to define a continuous map to the boundary of the disk, which is a retraction. By applying the fundamental group functor π_1 to the diagram

$$\begin{tikzcd} S^1 \arrow[r, hook] \arrow[dr, "\mathrm{id}"'] & D \arrow[d, "f"] \\ & S^1 \end{tikzcd}$$

we obtain a diagram of fundamental groups

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[dr, "\mathrm{id}"'] & * \arrow[d] \\ & \mathbb{Z} \end{tikzcd}$$

and this is simply impossible by very elementary group theory.

Thus the allure of algebraical methods is clear. However, when doing algebraic geometry, the spaces are often equipped with very unfavourable topological structures rendering standard algebraic topology uninteresting. Through generalisations of the concepts of topology and topological invariants, the work of Grothendieck and many other mathematicians in the previous century have improved the situation considerably.

In this thesis we will focus on the fundamental group through a series of reformulations that are not equivalent, but which have enough overlap to be interesting manifestations of similar underlying properties. In chapter 1 we will recall how the fundamental group relates to *covering maps* and look at some analogous properties from Galois theory. In chapter 2 we will describe how Grothendieck defined the *étale fundamental group* fully in terms of covering maps. This is done for schemes, and the covering maps are *finite étale maps*. For this basic theory, most of the content can be found in [Sza09], although we treat the subject slightly more generally. In chapter 3 we will see how this can be interpreted in a truly homotopy-theoretical setting, and how one can define higher homotopy groups through the *étale homotopy type*. The fundamental references for this are the books [AM86] and [Fri82]. In chapter 4 we will look at recent work in so-called *anabelian geometry*, an area driven by inspiration from classical topology. The fundamental conjectures of anabelian geometry were formulated by Grothendieck, and the paper [SS16] that we present here is some of the latest and most general proven positive results confirming them.

1.2. Covering theory in topology

Let us consider the classical theory of coverings in topology, to present the Galois correspondence that lies at the heart of this subject. Recall that the fundamental group is a functor

$$\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$$

taking a pointed space (X, x_0) to a group $\pi_1(X, x_0)$ of equivalence classes of loops, with concatenation of the representing loops as the group operation.

Definition 1.2.1. A *covering* of a space X is a pair $f : Y \to X$ of a space Y and a continuous function f such that f is a local homeomorphism, admitting a cover of open sets such that the preimage of such an open set U consists of disjoint open sets mapped homeomorphically to U. Thus the fibre of a covering map is a discrete set. The coverings of a space form a category, the subcategory of the slice category over X restricted to covering maps. A *universal covering* of X is a covering space that is simply connected. Such a covering is itself a covering for any other covering space of X, making it universal in the category of pointed covers of X.

Definition 1.2.2. The automorphism group $\mathrm{Aut}(Y/X)$ of the covering $p : Y \to X$ is the collection of covering morphisms $Y \to Y$ which is a group under composition.

Proposition 1.2.3 (Lifting property)**.** *Let X, Y and Z be locally connected and locally path-connected topological spaces. Covering maps of locally connected and locally path-connected spaces have a lifting property: for a continuous map from simply connected space Z there is (up to a choice of basepoint) a unique lifting to Y. In particular any path in X lifts to a path in Y.*

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ Z \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

For a pointed topological space X that is sufficiently nice (for instance connected, locally connected and locally simply connected) and a covering $p : Y \to X$, the fibre $p^{-1}(x)$ is a discrete set and the fundamental group $\pi_1(X, x)$ gives a left action on $p^{-1}(x)$ known as the monodromy action.

Definition 1.2.4. The *monodromy action* is defined in the following way: we begin by choosing an element $\alpha \in \pi_1(X, x)$ which is represented by loop $\gamma : I \to X$. By choosing a start point y above x, we get a lifting $\tilde{\gamma} : I \to Y$. In general this will not be a loop but a path from y to another point in the fibre over x, and this defines a permutation of the fibre.

Definition 1.2.5. A covering map $Y \to X$ where Y is locally path-connected and connected has a free action of $\mathrm{Aut}(Y/X)$ and if it is transitive we say that it is a *Galois covering*. In particular, every universal covering is Galois.

From this formalism we obtain powerful and fundamental results. Sending a cover $p : Y \to X$ to its fibre $p^{-1}(x)$ defines a functor Fib_x from the category of covers of X to the category of sets with a left $\pi_1(X, x)$-action.

Theorem 1.2.6 (Galois correspondence)**.** *Connected coverings $f : (Y, y) \to (X, x)$ give injections of the fundamental group $f_* : \pi_1(Y, y) \to \pi_1(X, x)$ by composing the representatives of the loops with f. There is a bijection between the fibre of x and the left cosets of $f_*(\pi_1(Y, y))$ in $\pi_1(X, x)$.*

Theorem 1.2.7 (Main theorem of covering theory)**.** *For a connected, locally connected and locally simply connected topological space X with base point x, the fibre functor* Fib_x *sending a covering p to the fibre $p^{-1}(x)$ is an equivalence of categories*

$$(\text{coverings of } X) \simeq (\text{left } \pi_1(X, x)\text{-sets}).$$

Connected coverings give sets with transitive action and Galois coverings give coset spaces of normal subgroups.

Remark 1.2.8. Recall that a functor on a category $\mathcal{C}$ is *representable* if it isomorphic to $\hom(c, -)$ for some object c of $\mathcal{C}$. One can verify that the fibre functor Fib_x is represented by the universal covering space, so that $\mathrm{Fib}_x(Y) = \hom_X(\tilde{X}_x, Y)$. This also implies that $\mathrm{Aut}\,\mathrm{Fib}_x \cong \mathrm{Aut}\,\tilde{X}_x \cong \pi_1(X, x)$.

Of course, for sufficiently nice spaces, there is always a universal covering space, which is simply connected and thus determines the fundamental group.

If we restrict ourselves to *finite covers*, i.e. covers $p : Y \to X$ where the fibres are finite sets, we get another equivalence of categories

$$(\text{finite coverings of } X) \simeq (\text{left } \widehat{\pi_1(X, x)}\text{-sets})$$

where the latter notation denotes the profinite completion of $\pi_1(X, x)$ (see the next section for details).

In algebraic geometry our spaces do not always have universal covers, but the analogy to this topological setting can be seen as a guiding principle of some of the constructions that are to be presented later on. For instance we can consider curves and construct a system of curves, a *pro-curve*, that behaves very much like a universal covering space. Since constructions like this are very important for what is to come, we will devote a section to the pro-prefix.

1.3. Infinite Galois theory

Recall that finite field extensions are algebraic and that an extension $K \subset L$ is Galois if $\mathrm{Aut}(L|K)$ fixes K and nothing else. Classical Galois theory tells us the following:

Theorem 1.3.1 (Main theorem)**.** *If $L|K$ is a finite Galois extension with Galois group G then the maps taking subfields M to $H = \mathrm{Aut}(L|M)$ and subgroups H to fixed fields $M = L^H$ is a bijection.*

Infinite Galois theory tells us that infinite Galois groups $\mathrm{Gal}(L|K)$ for Galois extensions $L|K$ are determined by finite quotients. To illuminate the similarity with covering theory the theorem can be reformulated in a more abstract way as follows.

Definition 1.3.2. A finite dimensional k-algebra A is *étale* if it is isomorphic to a finite product of separable extensions K_i of k,

$$A \cong \prod_{i=1}^{n} K_i.$$

Theorem 1.3.3 (Main theorem, modern version)**.** *For K a field, we have an anti-equivalence*

$$(\text{finite étale } k\text{-algebras}) \simeq (\text{finite continuous left } \mathrm{Gal}(k)\text{-sets})$$

by taking A to $\hom_k(A, k_{\mathrm{sep}})$ for a separable closure k_{sep} of k.

Here $\mathrm{Gal}(k) = \mathrm{Gal}(k_{\mathrm{sep}}|k)$ is the absolute Galois group of k viewed as a topological group, as in the definition and example below. Recall that a p-group is a group where every element has a finite order, which is a power of p. A purely group-theoretical fact is that if a p-group is finite it can't have a trivial centre.

Definition 1.3.4. A *profinite group* is a (topological) group that is an inverse limit of finite groups in the category of topological groups, where the finite groups are given the discrete topology. Similarly, a *pro-p-group* is a limit of finite p-groups.

Examples 1.3.5. Here are some profinite groups.

1. Finite groups are trivially profinite.
2. For any group G where we index its normal subgroups with the set Λ, we get canonical maps $G/H_\beta \to G/H_\alpha$ for α, β in $H_\beta \subset H_\alpha$. Defining the natural partial order on the indices we get an inverse limit $\widehat{G}$ called the *profinite completion* of G. The main example of such a group is $\widehat{\mathbb{Z}}$.
3. In the same vein, the ring of p-adic integers $\mathbb{Z}_p$ is a pro-p-group. There is also an isomorphism
$$\widehat{\mathbb{Z}} \cong \prod_{(0)\neq(p)\in \mathrm{Spec}\,\mathbb{Z}} \mathbb{Z}_p\,.$$

Examples 1.3.6. Some further examples of profinite Galois groups:

1. For $K|k$ Galois, the intermediate Galois extensions $k \subset M \subset K$ and $k \subset L \subset K$ with $L \subset M$ Galois, the Galois groups
$$\mathrm{Gal}(M|k) \to \mathrm{Gal}(L|k)$$
form an inverse system and we have
$$\mathrm{Gal}(K|k) \cong \varprojlim_M \mathrm{Gal}(M|k).$$
2. For every finite field $\mathbb{F}_q$ and a fixed separable closure, the algebraic and separable closures coincide: $\overline{\mathbb{F}_q} = \mathbb{F}_q^s$. Furthermore, there is for every n an extension $[\mathbb{F}_{q^n} : \mathbb{F}_q] = n$ with $\mathrm{Gal}(\mathbb{F}_{q^n}|\mathbb{F}_q) = \mathbb{Z}/n$. Thus we get that $\mathrm{Gal}(\mathbb{F}_q^s|\mathbb{F}_q) = \widehat{\mathbb{Z}}$.

3. For μ_{p^∞} the infinite group of p^n-roots of unity for all n, we have $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})|\,\mathbb{Q}) = \mathbb{Z}_p^\times$.

4. For μ the infinite group of all possible roots of unity for all n, we have $\mathrm{Gal}(\mathbb{Q}(\mu)|\,\mathbb{Q}) = \widehat{\mathbb{Z}}^\times$.

1.3.a. Pro-categories

Definition 1.3.7. A category is *cofiltering* (or *left filtering*) if for all pairs A and B of objects, there is a third object C for which there are maps

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

and if there are parallel arrows $A \rightrightarrows B$ then there is an arrow

$$C \longrightarrow A \rightrightarrows B$$

making the compositions equal, also called a *left equaliser.*

Example 1.3.8. The category of open subsets of a topological space is cofiltering.

Example 1.3.9. The category of connected pointed étale coverings of a pointed scheme is a cofiltering category.

Definition 1.3.10. A diagram $\mathcal{D} \to \mathcal{C}$ is *cofiltering* if $\mathcal{D}$ is a cofiltering category. We usually assume cofiltering categories to be small.

Definition 1.3.11. A functor $F : \mathcal{C} \to \mathcal{D}$ is *left final* if

1. For all d there is a c and a map $Fc \to d$.
2. For all c and every pair of maps $Fc \rightrightarrows d$ there is a map $c' \to c$ such that $Fc' \to Fc$ is a left equaliser.

Definition 1.3.12. Given a category $\mathcal{C}$, the *pro-category* pro-$\mathcal{C}$ is defined by taking objects all cofiltering diagrams in $\mathcal{C}$ and arrows

$$\hom_{\text{pro-}\mathcal{C}}(X, Y) = \lim_s \operatorname{colim}_t \hom_{\mathcal{C}}(X_t, Y_s).$$

An object of a pro-category is usually referred to as a pro-object and other terms are often given the pro- prefix in ways that are hopefully clear from context.

Examples 1.3.13. The most urgent example is that of profinite groups, which will get more attention below. Another example that appears in the literature is pro-$\mathcal{H}$, where $\mathcal{H}$ is the homotopy category of CW-complexes. We will devote more attention to these matters in chapter 3.

1.4. The étale fundamental group of curves

Here we present an elementary approach to fundamental groups for curves, which is a restriction of the general construction, but rich enough to be interesting and to give some intuition before turning to more general cases in the following chapters. This expostion mostly follows chapter 4 of [Sza09].

1.4.a. First encounters with étale maps

Let us get some insight into how the notion of étale k-algebras relates to curves and schemes. In scheme-theoretic language curves are much easier to define, but in this section we will do a scheme-free discussion of curves. Full details on these classical concepts can be found in chapter 4 of [Sza09] or alternatively in the first chapters in [Sil92] and [Har77] respectively.

Let k be a field. Recall that $\mathbb{A}_k^n$ is the affine n-space over k, consisting of all n-tuples of elements of k and that it corresponds to the ring $k[x_1, ..., x_n]$. For an ideal $I \subset k[x_1, ..., x_n]$ we have the affine closed set $X = V(I)$, which is given the Zariski topology and has the coordinate ring

$$\mathcal{O}_X(X) = k[x_1, ..., x_n]/I$$

which is reduced by definition. A surface defined by a single polynomial $I = (f)$ in $\mathbb{A}^2$ is called an *affine plane curve*.

An *integral affine curve* over a field k can be obtained from an integral domain A containing k which is finitely generated and of transcendence degree 1 over k. Every non-zero prime ideal of A is maximal and we set $X = \operatorname{Spec} A$, the topological space of all prime ideals of A. For a point $x \in X$ we have the local ring $\mathcal{O}_{X,x} = A_{P_x}$ where P_x is the prime ideal corresponding to x. For an open set U we define

$$O_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x} \,.$$

An integral affine curve is *normal* if all the local rings are integrally closed.

For k a field and $k \subset K$ a finitely generated extension of transcendence degree 1 we consider the set of discrete valuation rings containing k whose fraction field is K. This is an *integral proper normal curve* over k with function field K. The curve is given a topology where the closed proper subsets are the finite sets. For every open subset U we have a ring $\mathcal{O}(U)$ defined as the intersection of all rings $R \in U$. In fact, every integral proper curve can be covered by two integral affine curves, and the reader may thus recognise it as a scheme $(X, \mathcal{O}_X)$ with structure sheaf $\mathcal{O}_X$.

A morphism $Y \to X$ of integral affine curves is a continuous map ϕ such that any element $f \in \mathcal{O}_X(U)$ gives an element $g \in \mathcal{O}_Y(\phi^{-1}(U))$. Morphisms of integral proper normal curves are defined similarly. In the following, the word curve will refer either to an integral affine curve or an integral proper curve.

Definition 1.4.1. A morphism of curves $Y \to X$ is *separable* if the induced extension of function fields is a separable field extension. A morphism of affine integral curves is *finite* if $\mathcal{O}_Y(Y)$ is a finitely generated $\mathcal{O}_X(X)$-module via the induced map. The definitions of finite and separable morphisms carry over directly to integral proper normal curves.

Definition 1.4.2. For a finite separable morphism of integral affine curves $Y \to X$ we have a corresponding inclusion of rings $A \subset B$. We say that the morphism is *étale over* a closed point $P \in X$ if B/P is a finite étale A/P-algebra. The morphism is *étale* over an open set U if it is étale over all closed points $P \in U$. A map $\varphi : Y \to X$ of proper integral curves is étale at a closed point $P \in X$ if there is an open affine set $P \in U$ of X such that the restriction to $\varphi|_{\varphi^{-1}U}$ is étale in the previous sense over P.

Remark 1.4.3. For normal integral affine curves, the rings A and B are Dedekind rings, and we have a unique factorisation $PB = P_1^{e_1} \cdots P_m^{e_m}$. We also have inclusions of discrete valuation rings $\mathcal{O}_{X,P} \subset \mathcal{O}_{Y,P_i}$ and the morphism being étale at P is equivalent to that the powers e_i are all equal to 1 and that the extensions $k(P_i)|k(P)$ are separable. The condition that $e_i = 1$ is precisely that the map is *unramified* at P. These ideals correspond to the summands in the direct sum

$$B/PB = \bigoplus B/P_i^{e_i}B$$

and also to the points in the fibre above P. A simple example illustrates this.

Example 1.4.4. The map $\mathbb{A}^1 \to \mathbb{A}^1$ over $\mathbb{C}$ given by $z \mapsto z^n$ corresponds to a self-map of the ring $\mathbb{C}[x]$. A closed point gives a prime ideal of this ring of the form $P = (x - a)$. Let α_k be the nth roots of a. The quotient B/P is

$$\mathbb{C}[x]/(x-a) = \begin{cases} \bigoplus_{k=1}^{n} \mathbb{C}[x]/(x-\alpha_k) & \text{for } P \neq 0 \\ \mathbb{C}[x]/(x^n) & \text{for } P = 0 \end{cases}$$

which is clearly étale except for $P = 0$.

Example 1.4.5. If X is a geometrically integral affine curve over k and L is a finite separable field extension of k, then the base change $X_L \to X$ is finite and étale over every point of X. For a point $P \in X$ we have $\mathcal{O}(X_L)/P\,\mathcal{O}(X_L) \cong k(P) \otimes_k L$ which is a product of finite separable field extensions of $k(P)$.

In the theory of Riemann surfaces, one sometimes encounters étale as meaning local isomorphisms that are covering maps for some restriction. A well-known result saying that proper holomorphic maps of Riemann surfaces is a cover for all but a discrete set of points has an analogue:

Proposition 1.4.6. *A finite separable map $Y \to X$ of integral affine curves has a non-empty open subset U such that it is étale over all points of U.*

This is called a *finite branched cover*, and if the induced field extension of function fields $k(X) \subset k(Y)$ is Galois, it is a finite *Galois* branched cover.

Remark 1.4.7. For $K = \mathbb{C}$ we have deep connections to analytic geometry. There is a large body of work, notably the *GAGA* results by Serre, connecting analytical and algebraic sheaves. On a slightly more pedestrian level, we can still make some remarks. An integral proper normal curve X can be given a complex structure which gives a compact Riemann surface $X(\mathbb{C})$.

This constitutes a series of anti-equivalences of categories:

$$\left(\begin{array}{c}\text{integral proper normal}\\ \mathbb{C}\text{-curves with finite}\\ \text{morphisms onto } X\end{array}\right) \simeq \left(\begin{array}{c}\text{finite}\\ \text{extensions}\\ \text{of } k(X)\end{array}\right) \simeq \left(\begin{array}{c}\text{compact connected}\\ \text{Riemann surfaces with proper}\\ \text{holomorphic map onto } X(\mathbb{C})\end{array}\right)$$

1.4.b. First definition of the étale fundamental group

Proposition 1.4.8. *The category of finite field extensions $L|K$ is anti-equivalent to the category of integral proper normal curves with a surjective map $Y \to X$ where K is the function field of X. The inverse of the equivalence takes Y to its function field.*

This is proposition 4.4.5 of [Sza09].

Construction 1.4.9. If X is an integral proper normal curve over a perfect field k and U is a nonempty open subset, we consider the composite K_U of all finite subextensions L of K inside a fixed separable closure K_{sep} such that the corresponding curve is étale over all closed points P of U.

Proposition 1.4.10. *The extension $K_U|K$ is Galois and every finite subextension comes from a curve that is étale over U.*

Proof. This is proposition 4.6.1 of [Sza09]. If $L|K$ comes from a curve étale over U, then all subfields $L' \subset L$ does by lemma 4.5.10 of [Sza09]. If $M|K$ is another finite subextension that comes from a curve étale over U then so does the composite LM in K_{sep}. This is by the equality $\text{Gal}(K_{\text{sep}}|LM) = \text{Gal}(K_{\text{sep}}|L) \cap \text{Gal}(K_{\text{sep}}|M)$ from corollary 4.5.12 ibid. □

This allows a definition:

Definition 1.4.11. In the case above, we can define the *algebraic fundamental group* of U as

$$\pi_1(U) = \text{Gal}(K_U|K).$$

This construction depends on the choice of K_{sep}, which as we will see later acts as a base point.

Combining the two results 1.4.8 and 1.4.10 with some Galois theory gives the following result, which can be seen as a kind of Galois correspondence. In a sense that we will make precise in 2.3.1, this makes the left hand side category into a Galois category.

Theorem 1.4.12. *For the situation above, we have an equivalence of categories:*

$$\begin{pmatrix} \text{normal curves} \\ \text{finite separable étale over } U \end{pmatrix} \simeq \begin{pmatrix} \text{finite sets with} \\ \text{continuous left action by } \pi_1(U) \end{pmatrix}.$$

An obvious corollary gives a similar result for affine normal curves.

From the theory of Riemann surfaces, which is covered in [Sza09] and even more extensively in [For91], we know the following two propositions for the complex case.

Fact 1.4.13 ([Sza09] 3.6.3)**.** *Let X be obtained by removing r points from a connected compact Riemann surface of genus g. The (topological) fundamental group X is generated by elements $a_1, b_1, ..., a_g, b_g$ and $c_1, ..., c_r$ as*

$$\pi_1^{\text{top}}(X, x) = \langle a_i, b_i, c_j \mid \prod^g [a_i, b_i] \prod^r c_j = 1 \rangle$$

where $[f, g]$ is the usual commutator and x is some base-point.

Fact 1.4.14. *For X a connected compact Riemann surface and X' the surface after removing the r points, we have a composite field K of finite subextensions of $\mathcal{M}(X)$ that come from holomorphic maps $Y \to X$ for Y connected and compact which are unramified over X' (i.e. the r points are the ramification points), contained in some fixed algebraic closure. This is a Galois extension of $\mathcal{M}(X)$, and*

$$\text{Gal}(K|\mathcal{M}(X)) = \pi_1^{\text{top}}(X', x)^{\wedge}.$$

Combining 1.4.13 and 1.4.14 gives a useful result:

Theorem 1.4.15. *For X an integral proper normal curve over $\mathbb{C}$ with an open subset U with $\pi_1(U)$ the algebraic fundamental group defined as* $\text{Gal}(K_U|K)$*, we have isomorphisms*

$$\pi_1(U) = \pi_1^{\text{top}}(X', x)^{\wedge} = \langle a_i, b_i, c_j \mid \prod^g [a_i, b_i] \prod^r c_j = 1 \rangle^{\wedge}.$$

where X' is the Riemann surface associated to U.

Remark 1.4.16. A theorem that will be discussed later shows that if $k \subset L$ is an extension of algebraically closed fields of characteristic 0, the base-change functor $Y \mapsto Y_L$ is an equivalence of categories between finite étale coverings of X and those of X_L and thus there is an isomorphism $\pi_1(U_L) \cong \pi_1(U)$. In particular, curves over algebraically closed fields of characteristic 0 have fundamental groups given by the explicit presentation above.

Examples 1.4.17. For $k = \mathbb{C}$ (or any other algebraically closed field of zero characteristic) the results combine to show that $\pi_1(\mathbb{P}^1_k) = \pi_1(\mathbb{A}^1_k) = 1$ is the trivial group.

Similarly $\pi_1(\mathbb{P}^1_k \setminus \{0, \infty\}) = \hat{\mathbb{Z}}$. This shows in particular that for all $n > 0$ there is a unique isomorphism class of finite Galois coverings with group $\mathbb{Z}/n$ that are étale outside the two punctured points. These coverings comes from the coverings $x \mapsto x^n$ of the affine line $\mathbb{A}^1_k$.

Puncturing once more, we get that $\pi_1(\mathbb{P}^1_k \setminus \{0, 1, \infty\})$ is the free profinite group on two elements. Remarkably, since it can be shown that all finite simple groups are generated by two elements, they all arise as quotients of this group.

Remark 1.4.18. We illuminate how this relates to the universal covering of topological covering theory, which can be seen as a representing object for the fundamental group functor as in remark 1.2.8.

For a scheme X/K with the étale topology we can define a *pro-universal covering* as a pro-object, i.e. system of coverings, which will factor through any other cover.

That is, a pro-universal covering $\tilde{f} : \tilde{X} \to X$ is a system $(f_i : X_i \to X)$ of finite étale maps such that all finite étale coverings $g : Y \to X$ will have unique covering maps $X_i \to Y$ commuting with the maps above.

Denote the integral closure of $A = \mathcal{O}(U)$ in K_U by $\tilde{A}$. Suppose we have a finite subextension $K(V)$ of $K(X)$ in K_U that comes from a finite étale morphism $V \to U$. Then $\mathcal{O}(V) = \tilde{A} \cap k(V)$. A maximal ideal $M \subset \tilde{A}$ gives a closed point $M \cap \mathcal{O}(V)$. The ring $\tilde{A}$ is equal to the union of all finitely generated k-algebras $\mathcal{O}(V)$ for all such V and the set $\tilde{U}$ of maximal ideals therein is the inverse limit of the sets of closed points of these V.

This is endowed with an inverse limit topology and gives a ring $\mathcal{O}_{\tilde{U}}$. This construction can be viewed as a *pro-algebraic curve*, with *pro-points* $\tilde{Q}$ giving local rings $\mathcal{O}_{\tilde{U},\tilde{Q}}$ with pro-points at infinity lying over $X \setminus U$ and so on. This is a pro-étale covering analogue of the universal covering spaces that exist for (sufficiently nice) topological spaces.

Construction 1.4.19. For base fields that are not algebraically closed, a new set of properties arise. Let X be an integral proper normal curve over a perfect field k in some algebraic closure $\bar{k}$, and let K be the function field of X.

If $K \otimes_k \bar{k}$ is a field, i.e. when X is *geometrically integral*, then this field is the function field of the base-change $X_{\bar{k}}$. An affine open curve U gives an affine integral open curve $U_{\bar{k}}$.

Fix a separable closure K_{sep} containing $\bar{k}$ and K. In this closure the tensor product above can be identified with the composite $K\bar{k}$. It is also the composite of the function fields of all finite étale covers $X_L \to X$ with $L|k$ finite. The function field of the curve X_L is KL.

$$\begin{tikzcd} K \arrow[r, hook] & KL \arrow[r, hook] & K\bar{k} \\ k \arrow[r, hook] \arrow[u, hook] & L \arrow[r, hook] \arrow[u, hook] & \bar{k} \arrow[u, hook] \end{tikzcd}$$

The constructed field K_U is in K_{sep} and contains $K\bar{k}$, and the Galois group $\mathrm{Gal}(KL|K)$ is canonically isomorphic to $\mathrm{Gal}(L|k)$. As a result we have $\mathrm{Gal}(K\bar{k}|K) \cong \mathrm{Gal}(\bar{k}|k)$.

This discussion leads to this proposition, which is a special case of 2.3.33. We give a direct proof here.

Proposition 1.4.20. *For X geometrically integral proper normal curve over a perfect field k and $U \subset X$ open, we have an exact sequence*

$$1 \longrightarrow \pi_1(U_{\bar{k}}) \longrightarrow \pi_1(U) \longrightarrow \mathrm{Gal}(\bar{k}|k) \longrightarrow 1$$

Proof. We want to show that $\pi_1(U_{\bar{k}})$ is precisely the kernel. We can note that we have $X_{\bar{k}} \to X$ as a pro-finite étale covering, and K_U and $K_{U_{\bar{k}}}$ corresponding to the pro-coverings $\tilde{X}$ and $\tilde{X}_{\bar{k}}$ of Remark

1.4.18. Intuitively, we know that the finite coverings in the system defining $\tilde{X}_{\bar{k}}$ are finite coverings of X and thus are in the system defining $\tilde{X}$, and this will turn out to be true.

Let $G = \mathrm{Gal}(K_0|K\bar{k})$ be a finite quotient in the system defining $\mathrm{Gal}(K_{U_{\bar{k}}}|K\bar{k})$. Using the minimal polynomial for this extension, we may construct extensions $L|k$ and $L_0|KL$ with $\mathrm{Gal}(L_0|KL) = \mathrm{Gal}(K_0|K\bar{k}) = G$.

This L_0 corresponds to a covering $Y \to X_L$ which is étale over U_L and the base change $Y_{\bar{k}}$ is clearly the same as the covering defined by K_0. We see that $Y \to X_L \to X$ is a finite étale covering of X which is étale over U and so $L_0 \subset K_U$.

Since $L_0 \subset K_U$ we have shown that $K_{U_{\bar{k}}} \subset K_U$. We have extensions $K \subset K\bar{k} \subset K_U$ and so the kernel of the map from $\pi_1(U) = \mathrm{Gal}(K_U|K)$ to $\mathrm{Gal}(\bar{k}|k) = \mathrm{Gal}(K\bar{k}|K)$ is precisely $\mathrm{Gal}(K_U|K\bar{k})$ which thus surjects onto $\mathrm{Gal}(K_{U_{\bar{k}}}|K\bar{k})$.

Here we are using the following obvious property. The natural map from the Galois group of the left vertical arrow to the Galois group of the bottom arrow in

$$\begin{tikzcd}
K_U & K_{U_{\bar{k}}} \arrow[l, hook'] \\
K \arrow[u, hook] \arrow[r, hook] & K\bar{k} \arrow[u, hook'] \arrow[ul, hook']
\end{tikzcd}$$

has the one of the diagonal arrow as kernel, and the diagonal one surjects onto the right vertical one.

We have also shown that for every finite quotient G of $\pi_1(U_{\bar{k}})$, i.e. for every K_0 with Galois group G in the system defining $\pi_1(U_{\bar{k}})$ there is a finite quotient of $\mathrm{Gal}(K_U|K\bar{k})$, i.e. some L_0 with group G in the system defining $\pi_1(U)$. Every finite quotient of the former is mapped isomorphically to one of the latter and thus the surjection is an isomorphism, showing injectivity of the first arrow in the sequence.

An overview of the fields and the coverings can be put in a diagram:

$$\begin{tikzcd}
K_U & & K_{U_{\bar{k}}} \arrow[ll, hook', dashed] \\
 & L_0 \arrow[r, hook] & K_0 \arrow[u, hook, "\vdots"'] \\
K \arrow[uu, hook, "\vdots"] \arrow[r, hook] & KL \arrow[u, hook] \arrow[r, hook] \arrow[ur, phantom, "(1)"] & K\bar{k} \arrow[u, hook] \\
k \arrow[u, hook] \arrow[r, hook] & L \arrow[u, hook] \arrow[r, hook] & \bar{k} \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{X} \arrow[rr, dashed] \arrow[dd, "\vdots"'] & & \tilde{X}_{\bar{k}} \arrow[d, "\vdots"] \\
 & Y_L \arrow[d] & Y \arrow[l] \arrow[d] \\
X & X_L \arrow[l] & X_{\bar{k}} \arrow[l]
\end{tikzcd}$$

The proof consists mostly in constructing the square (1), noting that the two vertical arrows therein give the same Galois group, as well as using that the bottom two rows ($\bar{k}|k$ and $K\bar{k}|K$) have the same Galois group. The dots in the diagram indicates where we can identify a "limit process" (e.g. construction of a pro-cover) of "finite objects" (e.g. finite étale coverings). $\square$

Remark 1.4.21. (*Inner and outer automorphisms*) For a general short exact sequence of profinite groups

$$1 \to A \to B \to C \to 1$$

there is an action of B on the normal subgroup A by conjugation, and this gives a continuous homomorphism $B \to \mathrm{Aut}(A)$. The elements in $\mathrm{Aut}(A)$ that come from conjugation by an element of A makes up the group of inner automorphisms $\mathrm{Inn}(A)$. The quotient $\mathrm{Aut}(A)/\mathrm{Inn}(A)$ defines the group of outer automorphisms $\mathrm{Out}(A)$, and there is a continuous homomorphism $C \to \mathrm{Out}(A)$.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(A) & \longrightarrow & \mathrm{Aut}(A) & \longrightarrow & \mathrm{Out}(A) & \longrightarrow & 1 \end{array}$$

Remark 1.4.22. (*Outer Galois action*) Let as before X be a proper integral normal curve over a perfect field k with function field K. Assume that $K \otimes_k \bar{k}$ is a field. This is also the function field of the base change $X_{\bar{k}}$. For an affine open U we get the exact sequence of 1.4.20

$$1 \longrightarrow \pi_1(U_{\bar{k}}) \longrightarrow \pi_1(U) \longrightarrow \mathrm{Gal}(\bar{k}|k) \longrightarrow 1$$

The group $\pi_1(U_{\bar{k}})$ is called the *geometric fundamental group* of U. Constructing the inner and outer automorphism groups we get

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(U_{\bar{k}}) & \longrightarrow & \pi_1(U) & \longrightarrow & \mathrm{Gal}(\bar{k}|k) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \vdots & & \\ 1 & \longrightarrow & \mathrm{Inn}(\pi_1(U_{\bar{k}})) & \longrightarrow & \mathrm{Aut}(\pi_1(U_{\bar{k}})) & \longrightarrow & \mathrm{Out}(\pi_1(U_{\bar{k}})) & \longrightarrow & 1 \end{array}$$

which defines a natural continuous map

$$\rho_U : \mathrm{Gal}(\bar{k}|k) \to \mathrm{Out}(\pi_1(U_{\bar{k}})) = \mathrm{Aut}(\pi_1(U_{\bar{k}}))/\mathrm{Inn}((\pi_1(U_{\bar{k}}))$$

called the *outer Galois representation*, which is very important in number theory.

Chapter 2

The étale fundamental group

2.1. Algebraic geometry

Grothendieck initially formulated the ideas of sites, presented in section 2.2.1, to introduce a Grothendieck topology on schemes, or more precisely the *étale site* of a scheme. This section aims to introduce some concepts necessary for that construction, namely what a scheme is and what étale means.

Classical algebraic geometry was mainly concerned with studying the solution sets of polynomial equations through varieties, rings and other techniques from commutative algebra. The modern reformulation builds on Grothendieck's introduction of schemes, objects that are now at the center of attention in algebraic geometry.

In this section we will go through the basics of algebraic geometry, which is the framework where the general notion of étale fundamental group is defined. This material is well-established, and thus we will not spend too much time on it.

Assuming basic knowledge of commutative algebra and classical varieties at the level of [Rei95], we will start at the change of perspective. As is common to do, we will here require all rings to be commutative and unital. Good general references for schemes include [EH00] and [Har77]. A reader who is not satisfied with the brevity of these expositions might want to read the original source [GD71].

2.1.a. Basic notions

The following definition is central to the subject.

Definition 2.1.1. A *ringed space* $(X, \mathcal{O}_X)$ is a topological space X with a sheaf of rings $\mathcal{O}_X$ called the *structure sheaf*. A *morphism of ringed spaces* is a compatible pair of a continuous function $f : X \to Y$ on the space and a natural transformation $\mathcal{O}_Y \to f_* \mathcal{O}_X$ of the sheaves of rings. The *stalk* of a ringed space at a point is the stalk of the structure sheaf at that point.

Example 2.1.2. (*The ringed space* Spec A) Recall that a basis for the Zariski topology on $X = \operatorname{Spec} A$ is given by the principal open subsets of X, which are of the form $X_f = X \smallsetminus V(f)$. On this space we define a ring-valued sheaf by taking the basic open sets X_f to the localised rings A_f. The resulting sheaf is denoted $\mathcal{O}_X$ and makes $(X, \mathcal{O}_X)$ into a ringed space. The stalks of $\mathcal{O}_X$ are isomorphic to the localisations: $\mathcal{O}_{X,x} = A_P$ for P the prime ideal corresponding to $x \in X$. By [Har77] II.2.2. this sheaf is the unique sheaf on this space with these properties.

Definition 2.1.3. A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ whose stalks $\mathcal{O}_{X,x}$ are local rings. Morphisms of locally ringed spaces are morphisms of ringed spaces with the additional condition that the induced ring morphisms on the stalks are local.

Example 2.1.4. The sheaf Spec A is a locally ringed space. For a point x corresponding to a prime ideal P, the localisations A_f with $f \notin P$ form a system whose colimit is the local ring A_P which by the above example is the stalk $\mathcal{O}_{X,x}$.

Definition 2.1.5. An *affine scheme* is a locally ringed space isomorphic to Spec A for a ring A. A *scheme* is a locally ringed space that admits an open cover of affine schemes, that is, we can cover it by open sets of the form Spec A_i for rings A_i such that the restrictions of the structure sheaves agrees with the sheaves of the affine schemes.

Definition 2.1.6. The notion of a *morphism* of schemes is inherited from the category of locally ringed spaces. The category of schemes is denoted **Sch**. We often refer to objects of the slice category **Sch** $/S$ as schemes over S or S-schemes.

Remark 2.1.7. The mapping Spec is a contravariant functor from (commutative, unital) rings to affine schemes. It can be shown that it is full, faithful and essentially surjective, and thus an equivalence of categories. As $\mathbb{Z}$ is an initial object of the category of rings, $\operatorname{Spec}\mathbb{Z}$ is a terminal object in the category of affine schemes. Any scheme $X = \bigcup_i \operatorname{Spec} A_i$ has a unique morphism from every open set $\operatorname{Spec} A_i$ to $\operatorname{Spec}\mathbb{Z}$, which gives a morphism $X \to \operatorname{Spec}\mathbb{Z}$ which doesn't depend on the cover, which makes $\operatorname{Spec}\mathbb{Z}$ terminal for all schemes.

Remark 2.1.8. For the mappings Spec and Γ there is a natural isomorphism which is an adjoint relation

$$\hom_{\mathbf{Sch}}(Y, \operatorname{Spec} A) \cong \hom_{\mathbf{Rng}}(A, \Gamma(Y, \mathcal{O}_Y)).$$

Example 2.1.9. Some basic examples of schemes.

1. For any ring A we have the affine scheme $\operatorname{Spec} A$ as described above.
2. For k a field, the scheme $\operatorname{Spec} k$ has one point, and the stalk of that point is k.
3. For a discrete valuation ring R with fraction field K, the spectrum $\operatorname{Spec} R$ has a closed point for the maximal ideal of R, and a non-closed point for (0), the generic point. Thus the non-empty open subsets are the whole scheme and the complement of the closed point, which is (0). The rings of sections over these are R and K respecively.
4. For a finite étale k-algebra A, that is a finite product $A = \prod_{i=1}^{n} L_i$ of separable field extensions of k, the scheme $\operatorname{Spec} A$ is a disjoint union of points, one for every field in the product.
5. The projective line $\mathbb{P}^1_X$ as in example 2.1.38 below is a non-affine scheme.

Definition 2.1.10. The *dimension* of an affine scheme is the Krull dimension of the corresponding ring, and similarly for a general scheme it is the supremum of the length of chains of irreducible subsets. *Connected* and *quasi-compact* (that is, compact but not Hausdorff) schemes are defined to be schemes whose topological spaces satisfy the respective criteria.

Example 2.1.11. Let $X = \operatorname{Spec} A$ be an affine scheme. Suppose that we have an open covering $X = \bigcup_f X_f$ where $X_f = \operatorname{Spec} A_f$, then finitely many f suffice to cover. In other words, $\operatorname{Spec} A$ is quasi-compact.

Definition 2.1.12. A scheme is *reduced* if for all open subsets $U \subset X$, the ring $\mathcal{O}_X(U)$ has no nilpotents. Replacing nilpotents by zero-divisors, we have a definition of an *integral* scheme.

Fact 2.1.13. *A scheme X is integral if and only if it is reduced and irreducible as a topological space. ([Har77], II.3.1).*

Definition 2.1.14. The generic point of an affine integral scheme $\operatorname{Spec} A$ is the point corresponding to the zero ideal, and its closure is the whole space. Generally, any integral scheme has a unique generic point whose closure is the whole space.

Definition 2.1.15. A morphism $f : X \to Y$ is *of finite type at x* if x has an affine neighbourhood $U = \operatorname{Spec} B$ whose image is contained in some affine open $V = \operatorname{Spec} A$ such that the induced morphism $A \to B$ is of finite type, i.e. that B is isomorphic to a quotient of $A[x_1, ..., x_n]$ as an A-algebra. The morphism is *locally of finite type* if it is of finite type at all points.

Definition 2.1.16. A scheme is *normal* if the stalks $\mathcal{O}_{X,x}$ are integrally closed domains. It is *regular* if the stalks are regular local rings, where regular means that for the residue field κ and the maximal ideal m of the local ring A, we have $\dim_\kappa m/m^2 = \dim A$.

Fact 2.1.17. *A regular scheme is connected if and only if it is integral.*

Proposition 2.1.18. *The category* **Sch** *has binary fibre-products, or equivalently,* **Sch** */S has binary products.*

Proof sketch. The proof is by patching of the affine case, since the pushout of the anti-equivalent category of commutative rings $B \otimes_A C$ gives a pullback for affine schemes. A more rigorous and extensive proof can be found [GD71] Ch.I, 3.2.6. $\square$

Definition 2.1.19. For a morphism $X \to Y$ and a point y of Y corresponding to a prime ideal P of an affine open set $\operatorname{Spec} B = U$ in Y there are natural morphisms

$$B \to B_P \to \kappa(P)$$

and the composition of these gives a morphism $\operatorname{Spec} \kappa(P) \to U \subset Y$. The *fibre* of the morphism at the point y is defined as the pullback

$$X_y = X \times_Y \operatorname{Spec} \kappa(P).$$

Remark 2.1.20. This pullback does in general not agree with the pullback of the underlying topological spaces. For instance a separable field extension $k \subset L$ in a separable closure k_{sep} gives

$$\operatorname{Spec} L \times_{\operatorname{Spec} k} \operatorname{Spec} k_{\text{sep}} = \operatorname{Spec}(L \otimes_k k_{\text{sep}}) = \coprod_{i=1}^{n} \operatorname{Spec} k_{\text{sep}}$$

for some finite n but the underlying topological spaces are just points, and so the pullback is also a point:

$$\begin{array}{ccc} * & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

On the other hand, for a morphism $\phi : Y \to X$, the topological space of the fibre Y_P above a point $P \in X$ is homeomorphic to the topological preimage $\phi^{-1}(P)$.

Definition 2.1.21. An $\mathcal{O}_X$*-module* F on a scheme $(X, \mathcal{O}_X)$ is a sheaf of abelian groups on X such that $F(U)$ is an $\mathcal{O}_X(U)$-module for every U and such that the module operations commute with the restriction maps. A *sheaf of ideals* on a scheme $(X, \mathcal{O}_X)$ is an O_X-module I such that $I(U)$ is an ideal of $O_X(U)$ for every U.

Example 2.1.22. An A-module M defines a sheaf $\tilde{M}$ on $X = \operatorname{Spec} A$ by setting $\tilde{M}(D(f)) = M_f$. This is naturally an $\mathcal{O}_X$-module, and crucially $\Gamma(X, \tilde{M}) = M$.

Definition 2.1.23. An $\mathcal{O}_X$-module F on X is *quasi-coherent* if for every affine open $U = \operatorname{Spec} A$ of X the restriction $F|_U$ is isomorphic to the sheaf $\tilde{M}$ given by some A-module M.

Definition 2.1.24. For $X = \operatorname{Spec} A$ and an ideal J of A, we get $Y = \operatorname{Supp}(A/J) = V(J)$ as a closed subset of X. The pair $(Y, \mathcal{O}_{A/J}|_Y) = (\operatorname{Spec}(A/J), A/J)$ is a closed subscheme. This generalises to general schemes X by letting I be a quasi-coherent sheaf of ideals of $\mathcal{O}_X$ and setting $Y = \operatorname{Supp}(\mathcal{O}_X/I) \subset X$, as a closed subset of X. This gives $(Y, \mathcal{O}_X/I|_Y)$ as a *closed subscheme* of X. An *open subscheme* of X is a scheme induced on an open subset, and a *subscheme* is a closed subscheme of an open subscheme.

Example 2.1.25. There are distinct subschemes with the same underlying topological space. A basic example is that for any non-reduced ring A the reduction A_{red} is not the same ring, but has the same spectrum, $\operatorname{Spec} A = \operatorname{Spec} A_{\text{red}}$.

For a more general example we can take $(X, \mathcal{O}_X)$ as our scheme and then let N be such that $N_x = \operatorname{nilrad}(\mathcal{O}_x)$. We refer to the closed subscheme given by N as a *reduced* scheme and denote it by X_{red}. It is also called the reduction of X. Moreover, for any morphism $X \to Y$ there is a unique morphism $X_{\text{red}} \to Y_{\text{red}}$.

Example 2.1.26. Let A be a discrete valuation ring and let X be its prime spectrum. Any morphism $Y \to X$ has two fibres. The fibre over the closed point of the maximal ideal is called the special fibre, and it is a closed set. The fibre over the generic point is called the generic fibre, and it is open. Both the special and generic fibres can be empty.

Definition 2.1.27. An *immersion* is a morphism $Z \to X$ that factors through an isomorphism $Z \cong Y$ to a subscheme $Y \subset X$. An *open immersion* is one that factors through an open subscheme, and a *closed immersion* is one that factors through a closed one.

Example 2.1.28. Examples of immersions.

1. For an open subset $U \subset X$ we get an open subscheme $(U, \mathcal{O}_X|_U)$ and the inclusion morphism $U \hookrightarrow X$ is an open immersion.
2. For an affine scheme $\operatorname{Spec} A$, the morphisms corresponding to quotient morphisms $A \to A/I$ are closed immersions.

Definition 2.1.29. A morphism $f : X \to Y$ is *separated* if the diagonal morphism $\Delta : X \to X \times_Y X$ given by f is a closed immersion. For affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ the morphism comes from the surjection $B \otimes_A B \to B$ given by multiplication and is always separated. A scheme X is *separated* if its unique morphism $X \to \operatorname{Spec} \mathbb{Z}$ is separated. This is similar to how a topological space Z is Hausdorff if and only if the diagonal morphism $Z \to Z \times Z$ has closed image. Note that older terminology found in books such as [Mur67] and the first edition of [GD71] requires schemes to be separated and refers to what we call schemes as preschemes.

Fact 2.1.30. *Any morphism of affine schemes is separated.*

Lemma 2.1.31. *Let X and Y be schemes over S and let $S \to T$ be separated. Then the morphism $X \times_S Y \to X \times_T Y$ is a closed immersion.*

Proposition 2.1.32. *The following properties hold.*

1. *Immersions are separated.*
2. *If $X \to Y$ and $X' \to Y'$ are separated, then the induced $X \times_S X' \to Y \times_S Y'$ is.*
3. *Compositions of separated morphisms are separated.*
4. *Base-change preserves separatedness.*
5. *If $g \circ f$ is separated, then f is.*

Proof. We sketch the different items.

1. For closed immersions $f : X \to Y$ and an open $V \to Y$ the preimage $f^{-1}V \times_V f^{-1}V$ is open in the fibre product $X \times_Y X$. If a set of Vs cover Y then $f^{-1}V \times_V f^{-1}V$ cover the fibre product so we can reduce to the affine case by considering an open cover of Y. For the affine case we consider $\operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} A = \operatorname{Spec}(A \otimes_B A)$. If $B \to A$ is epi, then universal properties give directly that the multiplication morphism $A \otimes_B A \to A$ is an isomorphism, and thus surjective, so the morphism of affine schemes is a closed immersion.
2. This is a formal exercise in showing that $(X \times_S X') \times_{Y \times_S Y'} (X \times_S X') \cong (X \times_Y X) \times_S (X' \times_{Y'} X')$ which follows readily from the universal properties.
3. Let $X \to Y$ and $Y \to Z$ be separated. By considering

$$\begin{tikzcd} X \arrow[r, "\Delta_{X/Z}"] \arrow[dr, "\Delta_{X/Y}"'] & X \times_Z X \arrow[r] & Y \times_Z Y \\ & X \times_Y X \arrow[u] \arrow[r] & Y \arrow[u, "\Delta_{Y/Z}"'] \end{tikzcd}$$

the lemma gives that $\Delta_{X/Z}$ is a closed immersion since $\Delta_{Y/Z}$ and $\Delta_{X/Y}$ are closed immersions.

4. Consider $X' \to X' \times_{S'} X' \cong (X \times_S X) \times_S S'$ and apply (2) with $X \to Y$ and $X' = Y' = S'$.
5. Consider $X \xrightarrow{f} Y \xrightarrow{g} Z$ and the pullback diagram

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] & X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d] & Y \arrow[d, "g"] \\ & X \arrow[r, "gf"] & Z \end{tikzcd}$$

If gf is separated, (4) implies that π_Y is separated. We also have the base change

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"] & X \times_Z Y \arrow[d, "f \times 1_Y"] \\ Y \arrow[r, "\Delta_g"] & Y \times_Z Y \end{tikzcd}$$

and since Δ_g is an immersion, Γ_f is a immersion, and thus separated by (1). The composition $f = \pi_Y \Gamma_f$ is separated by (3).

□

Definition 2.1.33. A separated morphism $X \to Y$ is *proper* if it is of finite type and every base change $X \times_Y Z \to Z$ is a closed map.

Definition 2.1.34. A *finite* morphism is a morphism $X \to Y$ where Y has an affine covering $\operatorname{Spec} A_i$ whose preimages are affine sets $\operatorname{Spec} B_i$ with B_i finite A_i-modules.

Example 2.1.35. A finite morphism $f : X \to \operatorname{Spec} k$ for a field k has $f^{-1}(\operatorname{Spec} k) = \operatorname{Spec} A$ for a finite k-module and k-algebra A. In other words it is a ring morphism $k \to A$ such that A is a finite dimensional k-vector space.

Fact 2.1.36. *Any finite morphism and any closed immersion is proper. ([Har77] II.4.8, II.7). In fact, both finiteness and properness are stable by composition and base change.*

Fact 2.1.37. *A finite morphism of schemes is closed.*

Example 2.1.38. For $X = \operatorname{Spec} A$ we can consider the schemes $\operatorname{Spec} A[x]$ and $\operatorname{Spec} A[x^{-1}]$, both of which contain the subscheme $\operatorname{Spec} A[x, x^{-1}]$. Gluing the two schemes along this subscheme we obtain the projective line $\mathbb{P}^1_X$. The inclusion of A into $A[x, x^{-1}]$ gives a morphism $\mathbb{P}^1_X \to X$ which is proper. In general, we may construct the projective space $\mathbb{P}^n_X$ over any scheme X, and the natural morphism is proper, which follows from the case for $X = \operatorname{Spec} \mathbb{Z}$ and base change. See [Har77] for details.

Definition 2.1.39. A scheme is *locally Noetherian* if it can be covered by open sets $U_i = \operatorname{Spec} A_i$ where A_i are Noetherian rings. If the scheme can be covered by finitely many Noetherian rings, then it is *Noetherian.* If $X \to Y$ is locally of finite type and Y is locally noetherian, then it follows that X is locally noetherian as well.

2.1.b. Étale morphisms

Assume from now on that all schemes are locally noetherian and all morphisms locally of finite type.

Definition 2.1.40. We say that $f : X \to S$ is *unramified* at $x \in X$ if the image of the maximal ideal generates the maximal ideal $m_{f(x)} \mathcal{O}_x = m_x$, and if the extension of residue fields $k(x)/k(f(x))$ is finite and separable.

Example 2.1.41. Here is an example from number theory. Consider a number field K, a finite extension of $\mathbb{Q}$, for which we know that the ring of integers $\mathcal{O}_K$ is a Dedekind ring. This means that every ideal $I \subset \mathcal{O}_K$ has a unique factorisation in prime ideals

$$I = \prod p_i^{e_i}$$

where $e_i \in \mathbb{N}$ and $p_i \in \operatorname{Spec} \mathcal{O}_K$ are finitely many prime ideals. In particular a prime ideal p of $\mathbb{Z}$ is a subset of $\mathcal{O}_K$ and gives us an ideal $p\,\mathcal{O}_K$ which factorises in this way. If any of these powers e_i are larger than 1, we say that the extension $K|\,\mathbb{Q}$ is ramified at the corresponding prime p. If no primes are ramified, the extension is unramified.

Given a non-archimedean extension of local fields $L|K$, the situation is even clearer. The degree of the extension is $n = [L : K]$ and it can be shown that $n = ef$, where $f = [\lambda : \kappa]$ is the degree of the extension of residue fields and where e is the power in the equality $m_K \mathcal{O}_L = m_L^e$. A full exposition of number and local fields can be found in [Neu99].

Definition 2.1.42. A morphism of rings $A \to B$ is *flat* if it makes B a flat A-module. An A-module M is *flat* if the functor $M \otimes_A -$ preserves exact sequences. A morphism of schemes $f : X \to Y$ is *flat* if it induces flat morphisms on all stalks

$$f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x} .$$

Example 2.1.43. Field extensions are flat as vector spaces over a field. Free modules are flat. More generally, projective modules are also flat. A localisation $R \to S^{-1}R$ is a flat ring morphism and gives a flat morphism of schemes.

Definition 2.1.44. A morphism is *étale* if it is both unramified and flat.

Definition 2.1.45. An *étale covering* (French: *revêtement étale*) is a morphism that is étale and finite.

Remark 2.1.46. A particular feature of étale morphisms is that we can view a surjective family $\{U_i \to X\}$ as a single object $U = \coprod U_i \to X$ over X. Such an étale and surjective morphism is what in the next section will be regarded as a cover of X in the étale topology. Due to this, the terminology can sometimes be ambiguous. Some authors prefer to refer to étale coverings merely as *finite étale* morphisms, and covers of the étale topology (French: *recouvrement étale*) as *étale and surjective*, and the ambition of this report is to use those conventions as well.

Remark 2.1.47. Let $f : X \to Y$ be a finite morphism of curves. This gives a field extension of the fields of functions $[k(X) : f_* k(Y)]$. We can define the *degree* of the morphism as the degree of this field extension. More generally, we can define the degree of a finite étale morphism at a point $x \in X$. If it is constant we refer to it as the *degree of* f or $[X : Y]$. Some properties:

1. $[X : Y] = 0$ if and only if $X = \varnothing$.
2. $[X : Y] = 1$ if and only if f is an isomorphism.
3. $[X : Y] \geq 1$ if and only if f is surjective.

Examples 2.1.48. Some examples of étale maps.

1. The finite étale coverings of $\mathbb{G}_m$, the affine line $\mathbb{A}^1$ without the origin, over an algebraically closed field k of characteristic zero are the self-maps $t \mapsto t^n$. See also example 2.3.42.
2. A field extension $K \subset L$ gives a morphism $\operatorname{Spec} L \to \operatorname{Spec} K$ which is étale if and only if the extension is finite and separable.

3. The morphism given by inclusion $k \hookrightarrow L$ for an étale k-algebra $L = \prod_i K_i$, for K_i finite separable field extensions of k is étale.

Fact 2.1.49. *Some properties of étale maps:*

1. *Open immersions are étale.*
2. *Compositions of étale morphisms are étale.*
3. *Base-changes (of locally noetherian schemes) preserve étale morphisms.*
4. *If f_1 and f_2 are étale, then $f_1 \times_S f_2$ is.*
5. *If $g \circ f$ is étale and g is unramified then f is étale.*
6. *If $g \circ f$ is étale and f is étale and surjective then g is étale.*
7. *Étale morphisms are open.*

The last property gives that any finite étale morphism is both open and closed, and thus surjective in the connected, non-empty case.

Definition 2.1.50. A *variety* over a field k is an integral separated scheme of finite type over k.

Definition 2.1.51. A scheme X of locally finite type over a field k is *smooth over k* if after replacing k with $\overline{k}$, all the local rings are regular. In other words, the scheme has no singular points. A smooth scheme over a field is automatically regular and thus normal as well as reduced. Any smooth separated scheme of finite type over k is a finite disjoint union of smooth varieties over k.

These properties are fundamental for the definition of the étale site. Let us now recapture some further features of étale morphisms from [Mil13].

For a scheme with $x \in U = \operatorname{Spec} A$ an open affine neighbourhood, we have $\mathcal{O}_{X,x} = A_P$ where P is the prime ideal of A corresponding to the point x.

Definition 2.1.52. A *geometric point* for a scheme X is a morphism $\operatorname{Spec}\Omega \to X$ where Ω is a separably closed field. An *étale neighbourhood* of $\overline{x}$ is an étale morphism $U \to X$ with a geometric point $\overline{u}$ lying over $\overline{x}$.

$$\begin{tikzcd} U \arrow[r] & X \\ \operatorname{Spec}\Omega \arrow[u, "\overline{u}"] \arrow[ur, "\overline{x}"'] & \end{tikzcd}$$

Fact 2.1.53. *A connected étale neighbourhood has at most one morphism to another neighbourhood, and these form a directed system with $(U, u) \leq (U', u')$ if $U \to U'$ and $u \mapsto u'$.*

Definition 2.1.54. For this system we define the *local ring of the étale topology* as

$$\mathcal{O}_{X,\overline{x}} = \varprojlim \mathcal{O}_U(U).$$

As open immersions are étale, every Zariski open neighbourhood is an étale one, and we get a morphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\overline{x}}$.

Fact 2.1.55. *The local ring for the étale topology is a local noetherian ring with dimension equal to* $\dim X$. *It is also Henselian.*

Fact 2.1.56. *A morphism $\varphi : Y \to X$ is étale at y if and only if the ring morphism $\mathcal{O}_{X,\overline{\varphi(y)}} \to \mathcal{O}_{Y,\overline{y}}$ is an isomorphism.*

Proposition 2.1.57. *If a finite étale morphism $X \to S$ has a section, then this section gives an isomorphism of S and an open and closed subscheme of X. If S is connected, this is a connected component of X.*

Proof. The section $S \to X$ is étale and a closed immersion. Thus it is finite étale of degree 1 and is an isomorphism onto its image. Finite morphisms are closed and étale morphisms are open, and thus the image is closed and open. If X is connected then this must be the whole of X. □

Corollary 2.1.58. *Let X be connected. If two finite étale morphisms $X \rightrightarrows Y$ over S morphism a geometric point $\overline{x}$ to the same geometric points of Y over S, then the morphisms are equal.*

Proof. If the two morphisms both contain some geometric point their images are the same connected component and thus they agree on that component. □

A slightly more general variant is given in [Fri82], as proposition 4.1.

Proposition 2.1.59. *For a scheme X and an étale morphism $U \to X$ for a connected U, and an étale and separated morphism $V \to X$ the following holds. If f, g are two morphisms that agree on some geometric basepoint $u : \operatorname{Spec} k \to U$ as $fu = gu$, then $f = g$.*

$$\begin{tikzcd}
U \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr] & & V \arrow[dl] \\
& X &
\end{tikzcd}$$

2.2. Sites

2.2.a. Grothendieck topologies

Let us now return to more abstract matters by informally recalling the three properties that open covers of topological spaces satisfy:

1. Any open set is covered by itself.
2. If one covers every set in a cover, the collection of these covers will be a cover.
3. A given cover intersected with a subset will be a cover of the subset.

In his work Grothendieck formalised these properties abstractly. The resulting notions of Grothendieck topologies and sites replace topological spaces and enables us to use topological tools like sheaves in a wider range of situations. A good introduction to Grothendieck pretopologies is *Grothendieck Topologies* by M. Artin, [Art62].

Definition 2.2.1. For a category $\mathcal{C}$, a set J of covering families $\{\varphi_i : U_i \to U\}$ is called a *pretopology* if the following holds. For a given covering family, the codomain U is fixed and may be referred to as a *cover* of U. The set of covers should satisfy three axioms:

1. For any isomorphism φ, the set $\{\varphi\}$ is a cover.
2. For a cover $\{U_i \to U\}$ and for every i a cover $\{V_{i,j} \to U_i\}$, the composite family $\{V_{i,j} \to U\}$ is a cover
3. For any cover $\{U_i \to U\}$ and any $V \to U$ there are fibre products $U_i \times_U V$ and $\{U_i \times_U V \to V\}$ is a cover.

A collection J satisfying the first two of these is sometimes referred to as a basis for a Grothendieck topology or as *coverage*. In SGA4 and in modern works in topos theory we find the definition below. The functor $\hom(-, U)$ gives all maps into U, and this can be interpreted as all subobjects of U. Based on this observation, we introduce the notion of a *subfunctor*. For functors $F, G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ we say that G is a subfunctor of F if $G(c) \subset F(c)$ and if for all maps $f \in \mathcal{C}(c, d)$ we have that $Gf = Ff|_{Gd}$. A *sieve* of U is a subfunctor S of the functor $\hom(-, U)$. In the following definition, the notion of covering families is replaced by sets $J(U)$ of covering sieves of U.

Definition 2.2.2. A *Grothendieck topology* for a category $\mathcal{C}$ is a collection J of covering sieves defined as follows. $J(U)$ denotes the *covering sieves* of U, and these satisfy

1. The maximal sieve $\hom(-, U)$ is in $J(U)$.
2. For S in $J(U)$ and R any sieve of U, if for each $h : V \to U$ in S the pullback $h^*(R)$ is in $J(V)$, then R is in $J(U)$.
3. If S is in $J(U)$ and $h : V \to U$ then $h^*(S)$ is in $J(V)$.

Definition 2.2.3. A *site* is a category with a Grothendieck topology.

Remark 2.2.4. Some modern authors, such as [Mil13], instead define a site as a category with a pretopology. What SGA4 calls a pretopology is what Milne and the Stacks project calls a topology. These give the same categories of sheaves on the sites, but different notions of what it means to be local.

Remark 2.2.5. If $\mathcal{C}$ has fibre products we can generate a sieve F from a collection of arrows $\{f_i : U_i \to U\}$ by letting a map $h : V \to U$ be in $F(U)$ if it factors through some f_i as $h = f_i \circ \tilde{h}$. By this, we can define a pretopology from a topology by setting the covering families to be the ones that generates covering sieves. We verify that it is a pretopology.

Firstly, any isomorphism is a singleton covering family. Indeed, any isomorphism $V \to U$ can be used to factorise any map into U through V. Thus the generated sieve is $\hom(-, U)$ which is a covering sieve.

Secondly, if $\{f_i : U_i \to U\}$ generates a covering sieve S and if for each i the family $\{V_{i,j} \to U_i\}$ generates a covering sieve P_i, then the composite family generates a sieve R which should be a covering sieve. For any i, the pullback of the composite sieve R along f_i consists of all maps of R that factor through this particular f_i, and thus this pullback sieve is generated by the set of arrows $\{V_{i,j} \to U_i\}$, which gives P_i. Similarly, the third condition is also evident.

Conversely any pretopology gives a topology by setting the covering sieves to be all sieves that contain a covering family. However, this is not a bijection since several pretopologies can give rise to the same topology.

2.2.b. The étale site and other examples of sites

The discussion of the previous sections give the following.

Proposition 2.2.6. *The category Et/X of étale X-schemes, consisting of schemes Z with a fixed étale map $Z \to X$ has finite fibre products and a terminal object, X.*

Definition 2.2.7. A family of maps $\{\varphi_i : X_i \to X\}$ is *surjective* if X is the union of all $\varphi_i(X_i)$.

Definition 2.2.8. The *small étale site* X_{et} of a scheme X is the site given by Et/X and coverage given by surjective families. This gives natural notions of sheaves on X_{et} as well as the étale topos. More explicitly, the small étale site X_{et} is given as above by surjective families of étale maps, i.e. families $\{\varphi_i : X_i \to X\}$ such that φ_i are étale and X is the union of the images of the φ_i. If X is quasi-compact, it suffices with finite families of maps.

Remark 2.2.9. As stated before, in the étale site we can consider such a family $\{\varphi_i : U_i \to X\}$ as an object $\mathcal{U} = \coprod U_i$, with an étale map to X. That this always works is a property of the étale site that does not in general hold for other sites.

Remark 2.2.10. The abelian sheaf cohomology of an étale X-scheme X' is defined as the derived functor of the global section functor and denoted $H^q(X', F)$, giving us *étale cohomology*.

Example 2.2.11. Ignoring some set-theoretic problems, we can list some sites and facts about them.

1. The category of open subsets of a topological space is a site in the natural way.
2. The category of sets with covering families taken as surjective families of maps is a site.
3. The small étale site X_{et} is given as above by finite surjective families of étale maps $\{\varphi_i : X_i \to X\}$.
4. Our main example in the upcoming sections is the site X_{fet} given by surjective families of finite étale maps. The underlying category is also called $\mathbf{FEt}_X$ (for finite étale) and sometimes also Cov X (for étale coverings).
5. Similarly to the small étale site, the small Zariski site X_{zar} is given by surjective families of open immersions (instead of étale maps). This is a reformulation of the usual Zariski topology considered as a site.
6. The big étale site X_{Et} is given by the category $\mathbf{Sch}/X$ of schemes Z with a fixed but not necessarily étale map $Z \to X$ over X with covering families being surjective families of étale maps.
7. Similarly, there is a big Zariski site.
8. The Nisnevich site is given by surjective étale families such that for all u in U there exists some $i \in I$ and a $u_i \in U_i$ such that the map $k(u) \to k(u_i)$ is an isomorphism.

9. Let G be a profinite group and consider the category of finite discrete G-sets with covering families given by surjective families of G-maps. This is denoted T_G and is a site with what is called the *canonical topology*.

2.3. Galois categories and the fundamental group of a scheme

2.3.a. Galois categories

There are at least two ways of defining the étale fundamental group of a scheme. The approach taken by [Sza09] is the direct one: it is defined as the automorphisms of a particular functor, the fibre functor. The other approach is more general, but essentially the same. The construction is given a more abstract interpretation that indicates how the definition is the natural definition through the eyes of Galois theory. For that, we will now introduce the notion of a Galois category, as introduced in [Gro61], which can be thought of as a common generalisation of the properties exhibited in theorems 1.3.3 and 1.2.7.

Definition 2.3.1. A *connected object* in a category $\mathcal{C}$ is a non-initial object X such that if $Y \to X$ is a monomorphism for Y that is not an initial object, then $Y \cong X$.

Recall that a conservative functor is one that satisfies that if the image of a map is an isomorphism, then the map itself is an isomorphism. This can be seen as a surjectivity condition. It can also be referred to as that the map *lifts isomorphisms* or *reflects isomorphisms*.

Definition 2.3.2. A *Galois category* is a pair $(\mathcal{C}, F)$ with F a functor from $\mathcal{C}$ to finite sets called the *fibre functor* (or *fundamental functor*), such that $\mathcal{C}$ has all finite limits and colimits, and such that all objects of $\mathcal{C}$ are isomorphic to finite coproducts of connected objects. We require F to be conservative and exact (i.e. preserves all finite limits and colimits). The *fundamental group* $\pi_1(\mathcal{C}, F)$ is the automorphism group of the fibre functor.

Example 2.3.3. One example of a Galois category is the category of (finite continuous) G-sets with the forgetful functor to sets. This category has objects being sets equipped with a group action by G and maps being functions that are compatible with the actions. This forgetful functor is in fact represented by the group G, and the group G can be recovered from the functor as a kind of (Tannakian) reconstruction. The connected objects are the transitive G-sets and the usual orbit decomposition gives every object as a disjoint union of transitive sets. The theorem below shows that this is the *only* example, up to equivalence of categories.

Example 2.3.4. Let $S \in \mathbf{Top}$ be sufficiently nice to do classical covering theory. Let $\mathcal{C}$ be the category of finite covering spaces over S with maps being covering maps. For a fixed point $s \in S$ and a fixed covering $p : X \to S$ we consider the set $p^{-1}(s)$. This defines a functor by $(X, p) \mapsto p^{-1}(s)$ and gives a Galois category. In particular, S has a universal covering $\tilde{S}$ which gives $\pi_1(S, s)$ as the automorphism group on $\tilde{S}$ over S. The Galois correspondence of covering theory, Theorem 1.2.7, says that each connected object (X, p) of $\mathcal{C}$ determines a subgroup $H \subset \pi_1(S, s)$ up to conjugacy and each such subgroup H comes from a connected object (X, p). Also, $\hom(\tilde{S}, X) \cong p^{-1}(s) = F(X)$, meaning that F is a representable functor, as discussed in the introduction.

Example 2.3.5. Our main example of a Galois category in the sequel will be the category $\mathrm{Cov}\, X$ of finite étale maps over X for a geometrically pointed scheme X. Equivalently we can consider underlying category of the site X_{fet} where covers are given by surjective families of finite étale maps. The fibre functor is given by taking a scheme over X to the fibre of the basepoint. The definition of the étale fundamental group is thus easy to give: it is the fundamental group of the Galois category X_{fet}. In the following sections we will see what this means in more detail.

Remark 2.3.6. One might ask what happens if we drop the "finite" from "finite étale" requirement. This was explored by Grothendieck in SGA3 X.6, where he outlined the construction of an "enlarged" fundamental group. Leroy later implemented these ideas by studying Galois toposes, and showed a kind of Galois correspondence for these by sending a topos to the set-valued presheaves on its groupoid of points. Remarkably, the profinite completion of the enlarged fundamental group is isomorphic to the usual étale fundamental group.

One example of a Galois topos is the topos of locally constant sheaves on the small étale site. The automorphism group of one point of this topos is the enlarged fundamental group. On the other hand, considering the same construction on X_{fet}, the site where the maps also have to be finite, we get precisely the usual étale fundamental group. This will not be expanded upon further in this thesis, but it is of historical importance for the development of étale homotopy, which we will present in Chapter 3. Notably, the fundamental group of étale homotopy agrees with the enlarged fundamental group.

Remark 2.3.7. In [Gro61] (SGA1 V.5, def. 5.1) Grothendieck originally gave the definition of a Galois category as a category which is equivalent to π-sets for some profinite group π. From this he then derives a list of conditions G1-G6 on a category $\mathcal{C}$ with a functor F from $\mathcal{C}$ to finite sets that are equivalent. In fact, he also shows that there is a pro-object P that represents F. He refers to F as a fundamental functor of $\mathcal{C}$ and such a representing pro-object as a *fundamental pro-object*, and shows that the categories of fundamental functors and fundamental pro-objects are anti-equivalent. In particular, the fundamental functors of a Galois category $\mathcal{C}$ form a groupoid, the *fundamental groupoid* of $\mathcal{C}$, and all objects of this category gives the same automorphism group up to isomorphism.

Of course, these definitions agree. Let $(\mathcal{C}, F)$ be a Galois category. The theorem 4.4.1 of [Mur67], which is essentially a simplified special case of the results in [Gro61], states that:

Theorem 2.3.8. *For $\mathcal{C}$ a Galois category with fibre functor F, the following holds:*

1. *There exists a profinite group π such that F is an equivalence from $\mathcal{C}$ to finite π-sets with continuous action.*
2. *If F', π' gives another such equivalence, then $\pi' \cong \pi$ is an isomorphism in profinite groups canonical up to inner automorphism.*

Definition 2.3.9. An object in a Galois category is *Galois* if $X/\operatorname{Aut}(X) = *$. This means that the automorphism group of X acts transitively on X.

Example 2.3.10. In the category of G-sets, every open normal subgroup N gives a Galois object G/N. Indeed suppose G/H is Galois and let g be in G and let σ be an automorphism of G/H such that $\sigma(\overline{e}) = \overline{g}$ in G/H. Then for h in H we have $\overline{hg} = h \cdot \overline{g} = h \cdot \sigma(\overline{e}) = \sigma(\overline{h}) = \sigma(\overline{e}) = \overline{g}$ and thus $g^{-1}hg$ is in H, so H is normal, and conversely. The theorem implies that connected Galois objects in any Galois category corresponds bijectively to open normal subgroups of the fundamental group of the Galois category.

Definition 2.3.11. The *Galois closure* of a connected object X is a Galois object Y with a map $Y \to X$ such that for all Galois objects Z with maps $Z \to X$ the map factors through Y.

Proposition 2.3.12. *Any Galois category has Galois closures for all objects.*

Proposition 2.3.13. *If $(\mathcal{C}, F)$ is a Galois category and $\{X_i\}$ is a family of connected Galois objects such that every connected object in $\mathcal{C}$ is dominated by some X_i, then $\pi_1(\mathcal{C}, F) = \lim(\operatorname{Aut} X_i)^{\text{op}}$. That is, we can construct the fundamental group from the Galois closures.*

2.3.b. The Galois theory of schemes

Definition 2.3.14. The *fibre functor* of S_{fet} with base-point $\bar{s}$ is defined by $\operatorname{Fib}_{\bar{s}}$, taking a scheme X to the underlying set of $X_{\bar{s}} = X \times_S \operatorname{Spec} \Omega$. More succinctly, we may define it as the functor $- \times_S \operatorname{Spec} \Omega$ composed with the forgetful functor to sets.

We now consider the pair $(S_{\text{fet}}, \text{Fib}_{\bar{s}})$ as a Galois category for a scheme S with a geometric point $\bar{s}$. We need to show that the axioms hold for $(\mathcal{C}, F) = (S_{\text{fet}}, \text{Fib}_{\bar{s}})$.

Remark 2.3.15. If $Y' \to Y$ is a finite étale monomorphism in the category of schemes finite étale over X, then it is actually a monomorphism of schemes. It is an open immersion and finite, thus closed, and Y' is an open and closed subscheme. From this one can show that Y is connected in the sense above if and only if it is connected as a scheme, i.e. as a topological space.

Thus it follows that every object of the category is isomorphic to a finite coproduct of connected objects. The empty scheme is not connected, but it is the coproduct of zero connected schemes.

Proposition 2.3.16. *The category S_{fet} has finite colimits and limits.*

Proof sketch. To show that it is a Galois category we need to check that the category has finite limits and colimits.

For finite limits, it is enough to show that there is a terminal object and fibre products. For finite colimits, it is enough to check that there are finite coproducts and coequalisers.

The terminal object is the identity map $S \to S$. Similarly the initial object is the empty scheme $\varnothing \to S$ whilst coproducts and fibre products also are the ones one would expect.

The only non-trivial thing to show is the existence of coequalisers in this category. To show this, one could note that for a finite étale map $X \to S$ where s is a point of S, we can find an étale neighbourhood $(U, u) \to (S, s)$ such that $X_U = \coprod V_j$ is a finite disjoint union of sets V_j isomorphic to U containing all the points of the fibre of s. Looking locally at open sets around the points in the fibre, everything reduces to set maps between the index sets of the disjoint unions, which have coequalisers. With this perspective we could also say that any finite étale S-scheme is given by some finite locally free $\mathcal{O}_S$-algebra, and try to write out the constructions locally for rings that way. ☐

Proposition 2.3.17. *The fibre functor takes terminal objects to terminal objects.*

Proof. We consider the terminal object $S \to S$ and let $S_{\bar{s}}$ be the pullback of the identity and $\bar{s}$. For any $X \to S$ we consider $X_{\bar{s}}$, which fits into the diagram where Ω is abusive notation for $\operatorname{Spec} \Omega$, the field of the geometric point. By definition $S_{\bar{s}} = \operatorname{Spec} \Omega$ and

$$\begin{tikzcd}
X_{\bar{s}} \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & S_{\bar{s}} \arrow[r, equal] \arrow[d] & \Omega \arrow[d] \\
X \arrow[r] & S \arrow[r, equal] & S.
\end{tikzcd}$$

The dotted map exists and makes the combined diagram a pullback by universal properties and pullback pasting. Thus any geomtric fibre $X_{\bar{s}}$ has a unique map to $S_{\bar{s}}$ and this gives a unique map on the underlying set, thus this underlying set is terminal. ☐

It is equally trivial to see that the base change functor $S_{\text{fet}} \to S'_{\text{fet}}$ is exact for a map $S \to S'$. From this it follows that the fibre functor is exact.

Proposition 2.3.18. *For any map $f : X \to Y$ in S_{fet} we can factor it as $X \to Y' \to Y$ where the first map is epic and the second is monic and $Y = Y' \amalg Y''$, of which both are finite étale schemes over S.*

Proof. For $f : X \to Y$ set Y_1 to be the image. It is an open subscheme of Y. The pullback $X \times_S Y$ establishes $f \times 1$ as a section to the projection $X \times_S Y \to X$. This projection and the corresponding projection to Y are closed maps and so Y_1 is closed. By this, Y_1 is closed and open and thus $Y = Y_1 \amalg Y_2$ for some Y_2, also an open subscheme of Y. This gives the wanted factorisation. ☐

Proposition 2.3.19. *The fibre functor is conservative.*

Proof. If the map $\text{Fib}_{\bar{s}} X \to \text{Fib}_{\bar{s}} Y$ is a bijection then the degree of the covering is 1 everywhere and thus an isomorphism. ☐

Theorem 2.3.20. *The category of schemes finite étale over a scheme S is a Galois category.*

Proof. This is the content of the propositions above. $\square$

The abstract framework directly gives many nice properties which are inherited from the categories of G-sets through the equivalence of categories. We can now start stating some useful explicit details. To describe the Galois objects of our category we need some definitions.

Definition 2.3.21. For a S-scheme X we define $\operatorname{Aut}(X|S)$ as the group of scheme isomorphisms of X preserving the map to S. This is a group with a natural left action on X. By the base-change it also gives a natural left action on the fibre $X_{\bar{s}} = X \times_S \operatorname{Spec}\Omega$ of a geometric point $\bar{s}$ with field Ω.

Corollary 2.3.22. *For a connected finite étale map $X \to S$ only the identity element of $\operatorname{Aut}(X|S)$ has a fixed point.*

Construction 2.3.23. *For an affine surjective map $X \to S$ and a subgroup $G \subset \operatorname{Aut}(X|S)$ the map $X \to S$ factors through a morphism $\pi : X \to G\backslash X$, where the target is a scheme with structure sheaf $(\pi_* \mathcal{O}_X)^G$. In the affine case $X = \operatorname{Spec} B$ this is obtained by $G\backslash X = \operatorname{Spec} B^G$, and in the general case by patching. If the map $X \to S$ is a connected finite étale map and the group G is finite then both of these factors $X \to G\backslash X \to S$ are finite étale maps. The latter is locally due to $B^G \otimes_A C \cong (B \otimes_A C)^G$ and the former follows by base-change.*

Definition 2.3.24. A connected finite étale map is *Galois* if its automorphism group acts transitively on the geometric fibre.

These properties accumulate to the following proposition.

Proposition 2.3.25. *If X and Z are finite étale over S, where X is Galois, and $\pi : X \to Z$ is a morphism, then π is a finite Galois cover of $Z \cong H\backslash X$ for some subgroup H of $\operatorname{Aut}(X|S)$. This is a bijection:*

$$(\textit{subgroups}) \cong (\textit{intermediate coverings})$$

where the covering $Z \to S$ is Galois precisely when H is normal, and in this case $\operatorname{Aut}(Z|S) = \operatorname{Aut}(X|S)/H$.

2.3.c. The fundamental group

We consider the category S_{fet} and a geometric point $\bar{s} : \operatorname{Spec}\Omega \to S$ as before.

As usual, we define the fundamental group to be the automorphism group of the fibre functor,

$$\pi_1(S, \bar{s}) = \operatorname{Aut}(\operatorname{Fib}_{\bar{s}}).$$

Fact 2.3.26. *The fibre functor is pro-representable. There exists an inverse system $(P_\alpha, \phi_{\alpha\beta})$ of Galois coverings of S with $\operatorname{Fib}_{\bar{s}}(X) = \lim \hom_S(P_\alpha, X)$ for any X finite étale over S.*

The main theorem for this Galois category is the following slightly tautological proposition.

Theorem 2.3.27 (Main theorem). *Let $\bar{s}$ be a geometric point of a connected scheme S.*

1. *$\pi_1(S, \bar{s})$ is profinite and its action on every $X \in S_{fet}$ is continuous.*
2. *The fibre functor gives an equivalence of categories of S_{fet} and the category of finite continuous left $\pi_1(S, \bar{s})$-sets. Connected coverings correspond to sets with transitive action. Galois coverings correspond to finite quotients of the fundamental group.*

Proof. The main theorem can be shown directly: Choose an inverse system of Galois coverings P_α that pro-represents $\operatorname{Fib}_{\bar{s}}$. That the fundamental group is profinite follows directly from $\operatorname{Aut}(P_\alpha|S)$ being finite for every index α, and these automorphism groups gives an inverse system that induces the action on the profinite group. $\square$

Remark 2.3.28. We may recover the usual Galois theory of fields from the case $S = \operatorname{Spec} k$. A connected covering X is the spectrum of some separable field extension $L|k$ which gives $\mathrm{Fib}_{\bar{s}}(\operatorname{Spec} L) \cong \hom_k(L, k_{\mathrm{sep}})$ and thus $\pi_1(S, \bar{s}) = \mathrm{Gal}(k_{\mathrm{sep}}|k)$.

Example 2.3.29. (*The curve case*) For the previous case of curves, or more generally an integral normal scheme S with function field K in a separable closure K_{sep}, we can again let K' be the composite of the finite subextensions $L|K$ whose corresponding normalisation of S is an étale covering of S. As before $K'|K$ is Galois, and $\pi_1(S, \bar{s}) = \mathrm{Gal}(K'|K)$ for $\bar{s} : \operatorname{Spec} \overline{K} \to S$, meaning that our general definition really is a generalisation of the earlier definition. See chapter 5 of [Sza09].

Remark 2.3.30. For a sufficiently nice topological space Z with two points a, b and a path γ from a to b, we get an isomorphism $f : \pi_1^{\mathrm{top}}(Z, a) \to \pi_1^{\mathrm{top}}(Z, b)$ given by sending representing loops α to $\gamma^{-1} \cdot \alpha \cdot \gamma$. Suppose we have another path γ' giving another isomorphism f'. Then the first isomorphism is expressible in the second: $f = \gamma^{-1} \cdot \gamma' \cdot f' \cdot (\gamma')^{-1} \cdot \gamma$. This can be phrased thus: since the isomorphism depends on the choice of γ, the isomorphism is well-defined up to conjugation with an element $\gamma^{-1} \cdot \gamma'$. This is an *outer isomorphism*.

For a parallel discussion of schemes, let Z be a scheme and let $\bar{a}$ and $\bar{b}$ be geometric points. Then by definition we have $\pi_1(Z, \bar{a}) = \mathrm{Aut}(\mathrm{Fib}_{\bar{a}})$ and $\pi_1(Z, \bar{b}) = \mathrm{Aut}(\mathrm{Fib}_{\bar{b}})$. Instead of paths given by intervals, we have maps of fibre functors, defining $\pi_1(Z, \bar{a}, \bar{b}) = \mathrm{Isom}(\mathrm{Fib}_{\bar{a}}, \mathrm{Fib}_{\bar{b}})$. Any two elements γ and γ' here, they relate to each other as $\gamma' = \sigma_b \cdot \gamma \cdot \sigma_a$ for σ_i automorphisms of the corresponding fibre functor. Thus we have isomorphisms $\pi_1(Z, \bar{a}) \to \pi_1(Z, \bar{b})$ defined by $\tau \mapsto \gamma \cdot \tau \cdot \gamma^{-1}$, which is the same as $\eta \cdot \tau \cdot \eta^{-1}$ for $\eta = \sigma_a \cdot \gamma \cdot \sigma_b$.

Remark 2.3.31. In [Gro61] V Theorem 4.1 it is shown that every fibre functor is pro-representable by its path space $P_{\bar{a}}$, which is an element of pro-Cov X. As mentioned, or by Theorem 5.6 in SGA1.V, any two fibre functors are isomorphic and by [Sti13] 2.14 *any* map of fibre functors of a Galois category is an isomorphism. This opens up for a natural definition of a *fundamental groupoid* $\Pi_1(X)$ whose objects are fibre functors and whose arrows are transformations, *étale paths*, between them, $\pi_1(X; \bar{a}, \bar{b}) = \hom(\bar{a}, \bar{b})$. The étale fundamental group is recovered as the subcategory of automorphisms of a single point, $\pi_1(X, \bar{a}) = \hom(\bar{a}, \bar{a})$ just as the usual fundamental groupoid of a topological space.

2.3.d. Homotopy short exact sequence

When we have three Galois categories and two functors H, H' as in the sequence $\mathcal{C}_1 \to \mathcal{C}_2 \to \mathcal{C}_3$ we naturally get corresponding functors of G_i-sets for G_i the fundamental group of $\mathcal{C}_i$, which in turn gives maps h', h of profinite groups in a sequence $G_3 \to G_2 \to G_1$.

To arrive at an exact sequence of fundamental groups, we will look at some properties that a sequence of three profinite groups can have. Let us fix the notation above and consider some general categorical facts about Galois categories, as given in [dJea16]. They are all consequences of relatively straight-forward discussions of group and set theory.

Fact 2.3.32 ([dJea16])**.** *Some Galois-categorical facts.*

1. *The map $h : G_2 \to G_1$ is surjective precisely when H is fully faithful, which happens precisely when the existence of a map $* \to HX$ in $\mathcal{C}_2$ for X connected in $\mathcal{C}_1$ implies that X is terminal.*
2. *The map $h' : G_3 \to G_2$ is injective precisely when every X'' in $\mathcal{C}_3$ has some X' in $\mathcal{C}_2$ so that there is some Y'' in $\mathcal{C}_3$ with a monic map $Y'' \to H'X'$ and an epic map $Y'' \to X''$.*
3. *The composition hh' is trivial precisely when the image $H'HX$ is a finite disjoint union of terminal objects for all X in $\mathcal{C}_1$.*
4. *The group* $\operatorname{im} h'$ *is a normal subgroup of G_2 precisely when it is true that if some connected X' in $\mathcal{C}_2$ has a map $* \to H'X'$, then $H'X'$ is a finite disjoint union of terminal objects.*

5. *The map h is surjective and its kernel is the smallest normal closed subgroup that contains the image of h' precisely when conditions 1 and 3 are true, and if it is true that if $H'X'$ is a finite coproduct of terminal objects, then there is some X and some epimorphism $HX \to X'$.*

We will then consider the situation with $X_{\overline{k}} \to X \to \operatorname{Spec} k$, which gives us the sequence $\operatorname{Cov}\operatorname{Spec} k \to \operatorname{Cov} X \to \operatorname{Cov} X_{\overline{k}}$ of Galois categories, when X is geometrically connected, quasiseparated and quasicompact and give the homotopy exact sequence. The following theorem originally appeared in [Gro61] IX as Theorem 6.1.

Theorem 2.3.33 (Homotopy exact sequence). *For the situation $X_{\overline{k}} \to X \to \operatorname{Spec} k$ as described above when X is geometrically connected, quasiseparated and quasicompact, we get a short exact sequence*

$$1 \longrightarrow \pi_1(X_{\overline{k}}, \overline{x}) \longrightarrow \pi_1(X, \overline{x}) \longrightarrow \operatorname{Gal}(k) \longrightarrow 1.$$

Proof. ([dJea16] 7.1) The proof uses the fact 2.3.32 above.

1. For a covering of $\operatorname{Spec} k$, which is a finite separable field extension k' of k, there is a surjection $X_{\overline{k}} \to X_{k'}$ where the source is connected and thus the target is also connected.
2. For the second condition, we want to pick an object of $\mathcal{C}_3$ (a finite étale $Z \to X_{\overline{k}}$) and show that there is an object of $\mathcal{C}_2$ (a finite étale $Y \to X$) such that some object has a mono to $Y_{\overline{k}}$ and an epi to Z both finite étale over $X_{\overline{k}}$.

   Consider $Z \to X_{\overline{k}}$ finite étale and notice that the composition $Z \to X$ and $Z \to \operatorname{Spec} k$ fits in the diagram
$$\begin{tikzcd} Z \arrow[r] & X_{\overline{k}} \arrow[r] \arrow[d] & X \arrow[d] \\ & \operatorname{Spec}\overline{k} \arrow[r] & \operatorname{Spec} k. \end{tikzcd}$$

   Lemma 7.1 of [dJea16] uses that $\overline{k}$ is a colimit of all intermediate finite separable extensions $k'|k$, which implies that $X_{\overline{k}}$ is a limit of all $X_{k'}$. This then implies that given a finite étale map $Z \to X_{\overline{k}}$ there is some finite field extension k' and a finite étale map $Y \to X_{k'}$ such that $Z \cong Y \times_{X_{k'}} X_{\overline{k}}$. The composition $Y \to X_{k'} \to X$ is an object of $\operatorname{Cov} X$ with the desired properties since $Y \times_{X_{k'}} X_{\overline{k}} \to Y \times_X X_{\overline{k}}$ is a monomorphism.
3. A covering of $\operatorname{Spec} k$ gives a finite disjoint union of $X_{\overline{k}}$, which is the terminal object in the category of coverings of $X_{\overline{k}}$.
4. Here we choose a connected finite étale $U \to X$ and consider $H'U = U_{\overline{k}}$. A map from the terminal object is precisely a section $s : X_{\overline{k}} \to U_{\overline{k}}$. One can then consider
$$\bigcup_{\sigma \in \operatorname{Gal}(k)} s^{\sigma}(X_{\overline{k}})$$
   which is an open, closed and $\operatorname{Gal}(k)$-invariant subset of $U_{\overline{k}}$ and one can prove that this set is the inverse image of U, i.e. $U_{\overline{k}}$, and since the maps s^{σ} are isomorphisms (by $U_{\overline{k}}$ being connected and the sets being open and closed) we can identify $U_{\overline{k}}$ with a disjoint union of the terminal object $X_{\overline{k}}$.
5. For a finite étale $U \to X$ such that we can write
$$U_{\overline{k}} = \coprod X_{\overline{k}}$$
   there is some finite field extension k' of k such that
$$U_{k'} = \coprod X_{k'} = X \times_{\operatorname{Spec} k} \left(\coprod \operatorname{Spec} k' \right)$$
   and $U_{k'} \to U$ is the surjection we desire.

□

The theorem can also be proven directly, as is done in [Sza09]. This approach is perhaps more illuminating. The following couple of results are the chain of implications needed for that deduction as given by Szamuely.

Proposition 2.3.34. *Let S be connected. For two points $\bar{s}$ and $\bar{s}'$ with fields Ω and Ω' we have a (non-canonical) isomorphism $\pi_1(S, \bar{s}) \cong \pi_1(S, \bar{s}')$.*

Proof sketch. The proof relies on an isomorphism of functors $\mathrm{Fib}_{\bar{s}} \cong \mathrm{Fib}_{\bar{s}'}$ corresponding to two inverse systems $(P_\alpha, \phi_{\alpha\beta})$ and $(P_\alpha, \psi_{\alpha\beta})$. General results concerning Galois categories give that for each α there is a unique λ_α in $\mathrm{Aut}(P_\alpha|S)$ such that

$$\begin{tikzcd} P_\beta \arrow[r, "\lambda_\beta"] \arrow[d, "\phi_{\alpha\beta}"] & P_\beta \arrow[d, "\psi_{\alpha\beta}"] \\ P_\alpha \arrow[r, "\lambda_\alpha"] & P_\alpha \end{tikzcd}$$

that gives maps of sets $\mathrm{Aut}(P_\beta|S) \to \mathrm{Aut}(P_\alpha|S)$ which in turn gives an inverse system of finite sets, which has a non-empty limit, whose elements together amount to an isomorphism of fibre functors, which directly gives a continuous isomorphism of fundamental groups. □

Proposition 2.3.35 (Functoriality)**.** *For a morphism $\phi : S' \to S$ preserving basepoints $\bar{s}$ and $\bar{s}'$, there is a functor $B_{S,S'} = (- \times_S S')$ such that $\mathrm{Fib}_{\bar{s}} = \mathrm{Fib}_{\bar{s}'} \circ B_{S,S'}$ which thus gives a continuous map of profinite groups*

$$\phi_* : \pi_1(S', \bar{s}') \to \pi_1(S, \bar{s}).$$

A useful property for such a map is the following:

Proposition 2.3.36. *ϕ_* is trivial precisely when every base-change $X \times_S S'$ of a connected finite étale map $X \to S$ gives a trivial covering, and surjective precisely when every such base-change is connected. Here a trivial covering means that it is isomorphic to a finite disjoint union $\coprod S'$.*

A further characterisation of the image and kernel is the following.

Proposition 2.3.37. *For an open subgroup U of $\pi_1(S, \bar{s})$ the group $U \backslash \pi_1(S, \bar{s})$ corresponds to a connected cover $X \to S$ with a lifting $\bar{x} : \Omega \to X$ of $\bar{s} : \Omega \to S$. The following holds: U contains the image of ϕ_* if and only if the finite étale map $X \times_S S' \to S'$ has a section which maps $\bar{s}$ to $\bar{x}$. Similarly, replacing $U, S, \bar{s}$ by $U', S', \bar{s}'$ the following holds: U' contains the kernel if and only if there is some finite étale map $X \to S$ with some morphism over S' to X' from some connected component of $X \times_S S'$.*

With some profinite group theory, we get the following results.

Proposition 2.3.38. *ϕ_* is injective if and only if every connected finite étale cover $X' \to S'$ has some finite étale cover $X \to S$ as above.*

Corollary 2.3.39. *A sequence of pointed connected schemes*

$$(S'', \bar{s}'') \to (S', \bar{s}') \to (S, \bar{s})$$

gives a sequence

$$\pi_1(S'', \bar{s}'') \to \pi_1(S', \bar{s}') \to \pi_1(S, \bar{s})$$

that is exact if and only if (1) every finite étale map $X \to S$ gives a trivial covering $X \times_S S'' \to S''$, and (2) if every connected finite étale map $X' \to S'$ where $X' \times_{S'} S''$ has a section over S'' has some finite étale cover $X \to S$ and an morphism over S' from a connected component of $X \times_S S'$ to X'.

One more technical lemma remains. Let $k, \overline{k}$ and k_{sep} be as usual and let X be quasi-compact and geometrically integral over k, that is, $\overline{X} := X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ is integral.

Lemma 2.3.40. *A finite étale map $\overline{Y} \to \overline{X}$ has a finite étale map $Y_L \to X_L = X \times_{\operatorname{Spec} k} \operatorname{Spec} L$ where Y_L is the corresponding covering of a finite extension L of k in k_{sep}. This satisfies $\overline{Y} \cong Y_L \times_{\operatorname{Spec} L} \operatorname{Spec} k_{\mathrm{sep}}$. Also, elements of $\operatorname{Aut}(\overline{Y}|\overline{X})$ come from $\operatorname{Aut}(Y_L|X_L)$ for some finite extension L of k.*

The advanced reader will probably recognise that this latter statement is Galois theory in disguise, and will not be surprised that the proof is essentially ring-theoretical facts for rings A_i of affine open sets $\operatorname{Spec} A_i$ and verifications of compatibility along the intersection.

Now we give an alternative proof of the theorem.

Theorem 2.3.41 (Homotopy exact sequence)**.** *The sequence below is exact for X geometrically integral and quasi-compact with a geometric point $\overline{x}$.*

$$1 \to \pi_1(\overline{X}, \overline{x}) \to \pi_1(X, \overline{x}) \to \operatorname{Gal}(k) \to 1$$

Proof. (Szamuely 5.6.1) The corollary 2.3.39 and the lemma 2.3.40 above gives injectivity, and that X is connected gives surjectivity since every $Y \times_S S'$ will be connected for connected $Y \to X$.

The exactness in the middle is by the corollary 2.3.39 above. For (1): finite étale covers $X_L \to X$ with $L|k$ finite gives trivial coverings of $\overline{X}$. For (2): Pick a finite Galois $Y \to X$ where $\overline{Y} \to \overline{X}$ has a section. Then by lemma 2.3.40 there is some finite separable extension $k'|k$ such that the map $Y_{k'} \to X_{k'}$ has a section. This gives a finite étale cover $\operatorname{Spec} k' \to \operatorname{Spec} k$ and a morphism from $\operatorname{Spec} k' \times_k X = X_{k'}$ to Y. □

2.3.e. Some examples and calculations

Let us conclude this chapter with some examples.

Example 2.3.42. For an algebraically closed field k of characteristic zero, we can consider a finite étale map $Y \to X$ where $X = \mathbb{A}^1 - \{0\}$. This is then a discussion of smooth curves. As in [Sil92], we have a map of the projective completions $\overline{Y} \to \overline{X} = \mathbb{P}^1$ which is of the same degree n as the one above. This extension is of course finite étale with the possible exception of the two new points 0 and ∞. Riemann-Hurwitz gives that

$$2g - 2 = -2n + (e_0 - 1) + (e_\infty - 1)$$

where g is the genus of $\overline{Y}$ and e_i are ramification indices ranging over the all the points over 0 and ∞. If the number of points over 0 is a and over ∞ is b, then we have $2g - 2 = -a - b$, or in other words we have $2g + a + b = 2$ with $g \geq 0$, which forces $g = 0$ and $a = b = 1$. This means that $\overline{Y}$ is isomorphic to $\overline{X}$, and any étale map is of the form $t \to ct^n$. This shows that $\pi_1(\mathbb{A}^1_k - 0) = \widehat{\mathbb{Z}}$.

Remark 2.3.43. Example 2.3.42 is an algebraic derivation of something we already knew: by the comparison to the complex case, for which we can use analytic and topological methods, we already know what the fundamental group should be. The case where $\operatorname{char} k \neq 0$ however cannot be done analytically and is much more complicated. In fact, the structure of the group is not fully known, although much can be said about its prime-to-p part, where p is the characteristic. Let n be prime to p. From étale cohomology theory we can identify $H^1_{\mathrm{et}}(X, \mathbb{Z}/n\mathbb{Z})$ with finite étale maps $Y \to X$ with a free $\mathbb{Z}/n\mathbb{Z}$-action such that $X \cong Y/(\mathbb{Z}/n\mathbb{Z})$. Returning to the case of $\mathbb{A}^1 - \{0\}$, the coverings with a $\mathbb{Z}/n\mathbb{Z}$-action correspond to automorphisms sending $y \mapsto \zeta_n y$ where ζ_n is a primitive nth root of unity. Standard arguments of number theory shows that the number of primitive nth rooths of unity is determined by the Euler totient function. If k has nth roots of unity, we get that

$$H^1_{\mathrm{et}}(\mathbb{A}^1_k - \{0\}, \mathbb{Z}/n\mathbb{Z}) = \hom(\mu_n, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}.$$

This also shows that the fundamental group above is not merely $\widehat{\mathbb{Z}}$, but that it also has a "twisted" structure. By definition

$$\pi_1(\mathbb{A}^1_k - 0) = \lim_{\leftarrow n} \mu_n(k) = \widehat{\mathbb{Z}}(1)$$

is the *Tate twist*, i.e. $\widehat{\mathbb{Z}}$ with extra structure for the group actions by $\mathbb{Z}/n\,\mathbb{Z}$.

Example 2.3.44. Let us instead consider higher dimensions but keep k algebraically closed and of zero characteristic. A non-trivial result is that $\mathbb{A}^n - 0$ has no non-trivial finite étale covers for $n > 0$, and so $\pi_1(\mathbb{A}^n - 0)$ is trivial.

Example 2.3.45. Here are some standard examples.

1. For $X = \operatorname{Spec}\mathbb{Z}$, the maximal unramified extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$ is $\mathbb{Q}$ itself, and thus $\pi_1(X)$ is the trivial group.
2. There are no n-to-1 covers of $X = \mathbb{A}^1_{\mathbb{C}}$ for $n > 1$ by $2g - 2 \leq -2n + (n-1)$ (see [Sil92]) and so $\pi_1(X)$ is trivial.
3. For $X = \mathbb{A}^1_{\overline{\mathbb{F}_p}}$ there are many non-trivial unramified covers, for instance the self-map $x \mapsto x^p + x$. Thus $\pi_1(X)$ is not trivial.
4. For $X = \mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ we get

$$1 \to G \to \pi_1(X) \to \operatorname{Gal}(\overline{\mathbb{Q}}|\,\mathbb{Q}) \to 1$$

   where $G = \widehat{F}_2$ is the profinite completion of the free group on two elements. This gives a map $\operatorname{Gal}(\overline{\mathbb{Q}}|\,\mathbb{Q}) \to \operatorname{Out}(G)$ which is important in number theory.

Chapter 3

Étale homotopy

3.1. Homotopy theory

3.1.a. Introduction

In the previous chapter, we have seen how the covering theory approach led to a decent analogue for étale covers, but the story does not end there. In this chapter we will outline the contours of general homotopy theory and present the bare minimum needed to describe the Artin-Mazur, [AM86], and Friedlander, [Fri82], constructions of the étale homotopy type of a scheme.

In topology, the fundamental group forms a part of a larger picture of homotopy theory. As indicated by the notation, the functor π_1 is only a special case of the general formalism of homotopy groups π_n. The definitions are deceptively simple: for $n \geq 1$ the nth homotopy group of a pointed topological space X with basepoint x is defined as equivalence classes of basepoint-preserving continuous maps $S^n \to X$ modulo homotopy. For $n = 0$ we get a pointed set of path components, for $n = 1$ we get the fundamental group and for $n \geq 2$ we get the higher homotopy groups, which are always abelian.

In practice, these groups are rarely easy to compute, and thus homotopy theorists have developed many different abstract approaches and tools to handle them, rendering the subject fairly complicated. A core result is the following. A map of topological spaces $f : X \to Y$, $x \mapsto y$ induces morphisms $\pi_n(X, x) \to \pi_n(Y, y)$ for all n, which are homomorphisms for $n \geq 1$ and a mere function on $n = 0$. If these are isomorphisms for all n and all x, f is said to be a *weak homotopy equivalence*.

Theorem 3.1.1 (Whitehead 1949)**.** *A weak homotopy equivalence $f : X \to Y$ of CW complexes is a homotopy equivalence. Homotopy equivalences and weak homotopy equivalences coincide for CW complexes.*

As usual, for a map f to be a homotopy equivalence means that it has a homotopy inverse, which is a function $g : Y \to X$ such that both $g \circ f$ and $f \circ g$ are homotopic to the identity maps, which allows us to view isomorphisms in the homotopy category of topological spaces and maps that induce isomorphisms of homotopy groups (and sets) as equivalent. This desirable property is the basis for Quillen's introduction of model categories, an abstract formalism specifying three special kinds of morphisms: weak equivalences, fibrations and cofibrations, which gives a reasonable way to formally invert the weak equivalences and obtain a homotopy category. To give some relevant examples, we will now recall some basic facts about simplicial sets. A proper exposition of these matters can be found in standard references such as [GJ99].

3.1.b. Simplicial objects

Definition 3.1.2. The category of (set-valued) presheaves on a category $\mathcal{C}$ is denoted $\widehat{\mathcal{C}} = \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

Examples 3.1.3. Some trivial examples are that of the empty category $\mathbf{0}$, whose presheaf category is the singleton category, $\widehat{\mathbf{0}} \cong \mathbf{1}$. The presheaf category on $\mathbf{1}$ is isomorphic to the category of sets. For the category $\mathcal{C}$ of two objects $0, 1$ and two parallel arrows $0 \to 1$, the category $\widehat{\mathcal{C}}$ is isomorphic to the category of graphs. A presheaf F gives two sets $F(0)$ and $F(1)$, which can be interpreted as vertices and edges. The two functions $s, t : F(1) \to F(0)$ can be seen as assigning a source and a target to every edge. One example that generalises this is that of simplicial sets, which we treat on its own merits.

Consider the category $\mathbf{\Delta}$ of finite ordinals, i.e. objects $[n] = \{0 < 1 < ... < n\}$ and arrows being monotone maps $[n] \to [m]$. Here we have maps $d_i : [k] \to [k+1]$ where d_i hits every element except i, and maps $s_i : [k+1] \to [k]$ that hit i twice and all other elements once. The first few maps can be drawn in a diagram:

$$[0] \underset{d_0}{\overset{d_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{s_0}}}}} [1] \underset{d_0}{\overset{d_2}{\underset{\longrightarrow}{\overset{\longrightarrow}{\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{s_1}}}}}}} [2] \cdots$$

Fact 3.1.4. *Every map in* $\mathbf{\Delta}$ *is expressible as a composite of* d_i *and* s_i*-maps.*

This gives a generalisation of graphs: A presheaf $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ gives sets X_0, X_1... and maps between these corresponding to the d_i, s_i maps above. For instance, if $X_n = *$ for all $i \geq 2$ we get a graph.

Definition 3.1.5. The presheaf category $[\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C}]$ is denoted $S\mathcal{C}$ and its object are called *simplicial objects of* $\mathcal{C}$. The category of presheaves $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ is denoted $\widehat{\mathbf{\Delta}} = S\mathbf{Set} = \mathbf{ss}$ and the objects are called *simplicial sets*.

Using some category theory applied to Kan extensions, it is possible to describe many nice properties of the categories **ss** and **Top**. A pleasant and modern reference for this material is the first chapter of [Rie14]. The full machinery is not necessary for this thesis, but we indicate some of the implications:

Construction 3.1.6. Given $\Delta^\bullet : \mathbf{\Delta} \to \mathcal{E}$, where $\mathcal{E}$ is a cocomplete and locally small category and where we use the notation $\Delta^n = \Delta^\bullet([n])$ and for $X_. : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ we have

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, "y"] \arrow[d, "\Delta^\bullet"'] & S\mathbf{Set} \arrow[dl, dashed, "L=\mathrm{Lan}_y\Delta^\bullet"] \\ \mathcal{E} & \end{tikzcd}$$

where y is the Yoneda embedding. By the adjoint functor theorem we have a right adjoint $R : \mathcal{E} \to S\mathbf{Set}$ to L with $\mathcal{E}(\Delta^n, e) = (Re)_n$.

Example 3.1.7. For the inclusion $\mathbf{\Delta} \to \mathbf{Cat}$ where $[n]$ is mapped to the category given by the poset in the natural way by adding identities, we get a right adjoint which is commonly known as the *nerve* functor, and a left adjoint which maps a simplicial set to its homotopy category. Explicitly, the nerve of a category $\mathcal{C}$ is the simplicial set given by

$$N : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$$

given by $N(\mathcal{C})_n = \hom_{\mathbf{Cat}}([n], \mathcal{C})$, whose elements can be interpreted as strings of n composable arrows.

Example 3.1.8. (*Key example*) Consider

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r] \arrow[d] & \mathbf{ss} \arrow[dl, dashed] \\ \mathbf{Top}\,. & \end{tikzcd}$$

Here we get the adjoint pair S and $|-|$, which we can describe explicitly.

Denote the usual n-simplex realisation in $\mathbb{R}^n$ by Δ^n. For a topological space X there is a singular simplicial set S_*X where $S_iX = \hom_{\mathbf{Top}}(\Delta^n, X)$. This describes a functor S

$$\mathbf{ss} \underset{|-|}{\overset{S(-)}{\rightleftarrows}} \mathbf{Top}$$

which has an adjoint $|-|$ called *the geometric realisation* of a topological space. The functor $|-|$ can be described through the eyes of the Yoneda lemma in the following way. Any simplicial set is a colimit of its simplices,

$$X \cong \operatorname*{colim}_{\Delta^n \to X} \Delta^n$$

and the realisation functor can be described by

$$\begin{aligned} \mathbf{ss} &\longrightarrow \mathbf{Top} \\ \Delta^n &\longmapsto \Delta^n \\ X &\longmapsto \operatorname{colim} |\Delta^n| \end{aligned}$$

Example 3.1.9. Consider $\mathbf{\Delta}_n$, the subcategory of $\mathbf{\Delta}$ consisting of $[0], ..., [n]$, which gives a category of n-truncated simplicial sets $\mathbf{ss}_n$. The inclusion of $\mathbf{\Delta}_n$ into $\mathbf{\Delta}$ induces a functor i_* that has right and left Kan extensions:

$$\begin{tikzcd} \mathbf{ss} \arrow[r] & \mathbf{ss}_n \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}.$$

These functors are left and right adjoints of the inclusion functor, $i^* \dashv i_* \dashv i^!$. As usual, an adjoint pair $F \dashv G$ gives a monad GF and the monad defined by $i^! i_*$ is called the coskeleton and denoted cosk_n. The composite $i^* i_*$ is called the skeleton and denoted sk_n. These will be discussed further later on.

Definition 3.1.10. A *bisimplicial object* of $\mathcal{C}$ is a functor $X_{\bullet\bullet} : \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$. The terms $X(\Delta^m, \Delta^n)$ are denoted $X_{m,n}$. The *diagonal* of $X_{\bullet\bullet}$ is $\Delta X_{\bullet\bullet} = (X_{n,n})$.

3.1.c. Elements of simplicial homotopy theory

The data required for doing homotopy theory can be encoded in the requirements of a model category. A model category has three special classes of maps; the weak equivalences, the cofibrations and the fibrations. These yield subcategories which are closed under retracts, meaning that given

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "\mathrm{id}"] & X \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & Y \arrow[r] & B \end{tikzcd}$$

such that g is in one of the special subcategories, then f is also in that subcategory. Obviously all identity maps and isomorphisms qualify.

The weak equivalences are defined to have a 2-out-of-3-property: if $w = vu$ and two of the maps are weak equivalences, then the third one is also one. The fibrations and cofibrations are required to have certain lifting properties and are used to define *cofibrant* (and *fibrant*) objects; those for which the unique map from the initial object (respectively, to the terminal object) is a cofibration (fibration). Objects that are both are *bifibrant*. These special classes of objects have pleasant properties. For instance, if X is cofibrant, left homotopy is an equivalence relation on maps $X \to Y$ for any object Y, and a map between bifibrant objects is a homotopy equivalence if and only if it is a weak equivalence. From these classes of objects, one constructs the *homotopy category* of the model category, which is a localisation giving a formal inverse to every weak equivalence and identifying homotopic maps.

In the case of simplicial sets we can be explicit. The standard model structure defined by Quillen takes as fibrations the *Kan fibrations*, maps p that satisfy the property that every horn inclusion has a lift s as below (the catchy slogan is "*every horn has a filler*"):

$$\begin{tikzcd} \Lambda^m_n \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta_n \arrow[r] \arrow[ur, dashed, "s"] & B. \end{tikzcd}$$

A fibrant X is also called a *Kan complex*. Kan complexes have infinitely many simplices, and are generally hard to manipulate concretely and are better handled through abstraction. The similarity of this lifting property definition to that of Serre fibrations of topological spaces is not merely cosmetical: it is a theorem due to Quillen that a map f is a Serre fibration if and only if the simplicial map Sf is a Kan fibration. Also, the weak equivalences map to homotopy equivalences, and cofibrations are the monomorphisms. Furthermore the internal mapping space defined by $\mathrm{Map}(A, B)_n = \hom_{\mathbf{ss}}(A \times \Delta_n, B)$ gives a natural isomorphism $|\mathrm{Map}(A, B)| \cong \hom_{\mathbf{Top}}(|A|, |B|)$. Natural examples of Kan complexes are SX for any topological space X, as well as the nerve $N\mathcal{C}$ of any groupoid $\mathcal{C}$.

Informally, cofibrant objects has good structure on the maps mapping out of them, whilst fibrant objects have nice properties for mapping into them. In the standard model structure of topological spaces any topological space is fibrant, and examples of cofibrant spaces are CW-complexes, and in particular the spheres S^n. This is the reason why homotopy theory of topological spaces is well-behaved: the homotopy groups are defined in terms of maps from spheres, which are cofibrant, to topological spaces, which are fibrant.

Given a simplicial set X, we define $X \otimes \Delta^1$ by setting the nth level to $X_n \otimes \Delta^1_n$. There are two maps $e_0, e_1 : X \to X \otimes \Delta^1$ corresponding to the possible inclusions of the 0-simplex. A *strict homotopy* of maps $f, g : X \to Y$ is a map $h : X \otimes \Delta^1 \to Y$ where the composition $he_0 = f$ and $he_1 = g$. A *homotopy* is a chain of strict homotopies. Strictly speaking, we have defined a *left* homotopy, but every simplicial set is cofibrant. If Y is fibrant, model category theory gives that homotopy is an equivalence relation on maps $X \to Y$. The maps modulo homotopy is denoted $[X, Y]$.

It can be shown that for a simplicial set X, the natural map $X \to S|X|$ given from the adjunction is a weak equivalence. Moreover, any simplicial set is cofibrant and thus $S|X|$ is bifibrant. This shows that objects in general can be replaced by bifibrant objects up to weak equivalence, which is a central procedure in homotopy theory. A fibrant replacement functor which has better properties however, is the Kan replacement functor Ex^∞, which is obtained from the simplicial set $\mathrm{Ex}(X)_n = \mathbf{ss}(\mathrm{sd}\Delta^n, X)$ where sd is the barycentric subdivision. There is a natural map $X \to \mathrm{Ex}\, X$, and thus an infinite sequence $X \to \mathrm{Ex}\, X \to \mathrm{Ex}\,\mathrm{Ex}\, X \cdots$ and Ex^∞ is obtained as the colimit. Kan showed that it is a fibrant object, and that the map $X \to \mathrm{Ex}^\infty X$ is a natural weak equivalence, and for any X, Y we have isomorphisms

$$[X, Y] \cong [\mathrm{Ex}^\infty X, \mathrm{Ex}^\infty Y].$$

The skeleton and coskeleton functors are of great importance, and allows some degree of interpretation. For a simplicial set $X_\bullet$, the n-skeleton $\mathrm{sk}_n X_\bullet$ is the simplicial set that agrees with $X_\bullet$ up to dimension n and is trivial above it. The coskeleton is slightly more elusive: by adjointness,

$$(\mathrm{cosk}_k X_\bullet)_n = \hom_{\mathbf{ss}}(\Delta^n, \mathrm{cosk}_k X_\bullet) = \hom_{\mathbf{ss}}(\mathrm{sk}_k \Delta^n, X)$$

where the first equality is by the Yoneda lemma.

3.2. The étale homotopy type

3.2.a. The nerve theorem and hypercovers

Many methods and abstract tools relies on refinements, just as the Kan replacement functor can be seen as a colimit of better and better approximations. The Čech cohomology of Riemann surfaces is a somewhat accessible example:

Example 3.2.1. (*Low-dimensional Čech cohomology.*) Let F be a sheaf with values in the category of abelian groups, **Ab**. Let X be a topological space, for instance a Riemann surface. For a cover $\mathcal{U} = (U_i)_{i \in I}$ of X and an integer k, we define the kth cochain group of F as

$$C^k(\mathcal{U}, F) = \prod_{(i_l) \in I^{k+1}} F(U_{i_0} \cap \cdots \cap U_{i_k})$$

where the elements are called k-cochains. We have two coboundary operators

$$C^0(\mathcal{U}, F) \to C^1(\mathcal{U}, F) \to C^2(\mathcal{U}, F)$$

defined by mapping (f_i) to $(f_j - f_i)$ and (f_{ij}) to $(f_{jk} - f_{ik} + f_{ij})$. They are group homomorphisms and their composition is zero. Thus they form a cocomplex, and the first cohomology group is denoted $H^1(\mathcal{U}, F)$. With a bit of work, one can show that inclusions into finer covers give morphisms of cohomology groups, and we define the direct limit over all covers of X as the first Čech cohomology group of X,

$$\varinjlim_{\mathcal{U}} H^1(\mathcal{U}, F) = \check{H}^1(X, F).$$

The general idea can be described abstractly.

Example 3.2.2. Let $(U \to X) = \{U_i \to X\}$ be a surjective family in a site $\mathcal{C}$ with pullbacks. Let $\mathcal{U} \times_X \mathcal{U} = \{U_i \times_X U_j\} = \{U_{ij}\}$ be all double fibre products, $\mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U}$ be the triple fibre products and so on. We define a cocomplex $C^*(\mathcal{U}; F)$ by

$$C^n(\mathcal{U}; F) = \prod H^0(U_{i_0, \cdots, i_n}; F)$$

ranging over all possible n-tuples of indices. From this complex we get cohomology groups, and by taking a colimit over all possible surjective families $\mathcal{U}$ we obtain the Čech cohomology groups $\check{H}^n(X; F)$. In special cases this is isomorphic to the sheaf cohomology $H^n(X; F)$.

Construction 3.2.3. (*Čech nerve*) Let $\mathcal{U} \to X$ be as above. Then let $C(U)_\bullet$ be a simplicial object defined by $C(U)_n = \mathcal{U}^{\times_X^{n+1}}$, the $(n+1)$-fold fibre product over X. The face and degeneracy maps are natural: for instance we have maps $U_i \times_X U_j \to U_i \times_X U_i \times_X U_j$ and maps $U_i \times_X U_j \to U_i$ for all i and j.

Let us first review some properties of simplicial schemes, the simplicial objects in the category of schemes, or in our case, locally Noetherian schemes.

Example 3.2.4. For a scheme X and a simplicial set $T_\bullet$ we can set $(X \otimes T_\bullet)_n = \coprod_{t \in T_n} X$ with natural maps $(X \otimes T_\bullet)_n \to (X \otimes T_\bullet)_m$ given by the identity on X and the structure maps $T_n \to T_m$. This is a simplicial scheme. In particular, we can view every scheme as a simplicial scheme $X \otimes \Delta^0$, which is just X in every dimension.

Example 3.2.5. For a scheme S, a map $G \to S$ is a group scheme over S if $\hom_S(-, G)$ is a group valued functor on **Sch** $/S$. We can construct a classifying simplicial scheme BG by $BG_n = G^{\times_S^n}$, the n-fold fibre product.

Example 3.2.6. As in the simplicial set case, we have the coskeleton of a simplicial scheme. In particular if $X_\bullet$ is a simplicial scheme over a scheme S and $X_\bullet^{(n)}$ is the truncation, the universal property characterises $\operatorname{cosk}_n^S X_\bullet$ as thus: for any $Y_\bullet \to \operatorname{cosk}_n^S X_\bullet$ over S there is a natural bijection with maps $Y_\bullet^{(n)} \to X_\bullet^{(n)}$ over S. Of course, the truncation and coskeleton functors are (left respectively right) adjoints of each other.

Definition 3.2.7. For a scheme X over S, $\operatorname{cosk}_0^S X = \operatorname{cosk}_0^S(X \otimes \Delta^0)$ is *the Čech nerve $N_S(X)$.*

Here is a general result:

Theorem 3.2.8 (Nerve theorem, Borsuk 1948)**.** *Let X be a compact (or even paracompact) topological space, and let $\mathcal{U}$ be a cover such that all intersections U_{ij} et cetera are contractible, i.e. homotopy equivalent to a point. From the Čech nerve*

$$C(\mathcal{U}) = \left(\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \prod U_{ij} \rightrightarrows \prod U_i \right)$$

we can contract every set and get a simplicial set

$$\check{C}(\mathcal{U}) = \left(\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \prod * \rightrightarrows \prod * \right)$$

whose topological realisation $|\check{C}(\mathcal{U})|$ is homotopy equivalent to X.

However, this contractibility condition does not work well in general, or for the étale site in particular. In this setting, the Čech construction is not fine enough to capture all important information, or flexible enough to let us adjust for it, and so we need another approach. A suitable generalisation introduced by Verdier to solve this is the notion of hypercover. The idea is to instead of using double intersections, triple intersections and so on, we let every level be an open cover of the previous, such that $\mathcal{U}_0$ is a cover of X, $\mathcal{U}_1$ is a cover of the double intersections of $\mathcal{U}_0$ and $\mathcal{U}_2$ is a cover of the triple intersections of $\mathcal{U}_1$ and so on.

Construction 3.2.9. Let $\mathcal{U} \to X$ be a(n étale) cover. We start defining a simplicial object by letting $\mathcal{U}_0 = \mathcal{U}$. Let $\mathcal{U}_1$ be a cover of $\mathcal{U}_0 \times_X \mathcal{U}_0$. Here we use that $\mathcal{U}_0 \times_X \mathcal{U}_0 = (\operatorname{cosk}_0 \mathcal{U}_0)_1$. Let $\mathcal{U}_2$ be a cover of $(\operatorname{cosk}_1 \mathcal{U}_1)_2$ and so on. This process inductively gives a hypercover.

Definition 3.2.10. Let X be a scheme and consider the category of étale X-schemes. A *hypercover* is a simplicial object $\mathcal{U}_\bullet$ of this category such that $\mathcal{U}_0 \to X$ is an étale surjective map and $\mathcal{U}_{n+1} \to (\operatorname{cosk}_n \mathcal{U}_\bullet)_{n+1}$ is an étale surjective map for every $n \geq 0$.

Remark 3.2.11. The definition also works for any other site by replacing the covering families. For instance on the site of sets, if X is a point, the hypercoverings of X end up being contractible Kan complexes. (See 8.5 in [AM86] for this fact).

Example 3.2.12. The Čech nerve is an example of a hypercovering.

Remark 3.2.13. To provide the definition with some intuition, we can start by noting that the Yoneda lemma together with the adjointness of cosk_n and sk_n gives that

$$(\operatorname{cosk}_n Y)_k = \hom_{\mathbf{ss}}(\Delta^k, \operatorname{cosk}_n Y) = \hom_{\mathbf{ss}}(\operatorname{sk}_n \Delta^k, Y)$$

or in other words that we are interested in maps from the skeleton of a simplex to our object. For instance $(\operatorname{cosk}_0 \mathcal{U}_0)_1 = \hom_{\mathbf{ss}}(\operatorname{sk}_0 \Delta^1, \mathcal{U}_0)$ where $\operatorname{sk}_0 \Delta^1$ is the set of two points. Thus a map in the set is a choice of two sets $U_i \to X, U_j \to X$ in $\mathcal{U}_0$. Any map to these two elements factors through their fibre product U_{ij}, and so this provides us with a cover of $\mathcal{U}_0 \times_X \mathcal{U}_0$. Similarly for $(\operatorname{cosk}_1 \mathcal{U}_1)_2 = \hom_{\mathbf{ss}}(\operatorname{sk}_1 \Delta^2, \mathcal{U}_1)$, we know that $\operatorname{sk}_1 \Delta^2$ is the complex given by three points and three edges connecting them in a triangle, the boundary of Δ^2. A map from this into $\mathcal{U}_1$ corresponds to choosing three sets of $\mathcal{U}_1$ and compatible maps, and covering this set amounts to covering all triple intersections of $\mathcal{U}_1$.

The following result is from [AM86], corollary 8.13.

Fact 3.2.14. *The hypercoverings of a site form a category, with maps inherited from being simplicial objects. In the case of the étale site, it is a subcategory of the category of simplicial schemes étale over X. This category is not cofiltering. However, the category $HR(X)$ defined as hypercoverings modulo simplicial homotopy is cofiltering.*

The fact that it is cofiltering directly gives a pro-object of the homotopy category of simplicial sets (or equivalently, of CW-complexes).

Recall from the discussion of Galois categories that an object is connected if it is not the initial object and if it has no non-trivial coproduct decomposition. A category is called *locally connected* if any object is a coproduct of connected objects. Mapping an object to its set of connected components is a functor, denoted $\pi : \mathcal{C} \to \mathbf{Set}$.

Definition 3.2.15. A *point of a site* is an exact functor $p : \mathcal{C} \to \mathbf{Set}$ sending covering families to surjective families. A scheme pointed by a geometric point naturally gives a pointed étale site, where the point is the fibre functor.

Definition 3.2.16. A scheme is locally connected if its étale site is.

Fact 3.2.17. *A locally Noetherian scheme is locally connected. Every Y étale over X is a coproduct of connected schemes.*

Proof. Compare to the discussions in section 2.1, or see [GD71] I.6.1.9 or [AM86] chapter 8. □

Definition 3.2.18. The functor $\pi : \mathcal{C} \to \mathbf{Set}$ extends to a functor $S\mathcal{C} \to \mathbf{ss}$, and by taking quotients with simplicial homotopy we obtain a functor $\mathrm{Ho}(S\mathcal{C}) \to \mathrm{Ho}(\mathbf{ss})$ which naturally restricts to a functor $\pi : HR(\mathcal{C}) \to \mathrm{Ho}(\mathbf{ss})$ mapping a locally connected site $\mathcal{C}$ to the pro-object $\{\pi(K_\bullet)\}$ in pro-Ho($\mathbf{ss}$) where $K_\bullet$ is indexed over $HR(\mathcal{C})$.

Likewise, if the site is pointed we get an object of pro-Ho($\mathbf{ss}_*$). We can define pro-groups naturally by

$$\pi_q(\mathcal{C}) = \{\pi_q(\pi(K_\bullet))\}.$$

Remark 3.2.19. The assignment defined by taking $\mathcal{C}$ to the pro-object $\{\pi(\mathcal{C})\}$ is functorial with respect to morphisms of sites and is sometimes called the *Verdier functor*, but this name will not see extensive use in the sequel.

Definition 3.2.20. For X a locally Noetherian scheme we have a functor $HR(X) \to \mathrm{Ho}(\mathbf{ss})$ defined by taking a hypercovering $\mathcal{U}_\bullet$ to $\pi_0(\mathcal{U}_\bullet)$. This functor is a cofiltering diagram and thus defines a pro-object of $\mathrm{Ho}(\mathbf{ss})$ called X_{et}, the *étale homotopy type* of X. The pointed version of the construction is obtained in the obvious way.

We will not focus on the cohomology as we have our eyes set on the fundamental group, but an interesting result regarding the cohomology is the following, which is [AM86] Thm. 9.3.

Fact 3.2.21. *For an abelian group A, we naturally have a cohomology pro-group $H^n(X_{\mathrm{et}}, A)$. This is an abelian group and there is a canonical isomorphism of this to the étale cohomology group,*

$$H^n(X_{\mathrm{et}}, A) = H^n_{\mathrm{et}}(X, A).$$

3.2.b. The fundamental group of the étale homotopy type

Definition 3.2.22. A locally Noetherian scheme is *geometrically unibranch* if the integral closures of the local rings $\mathcal{O}_{X,x}$ are local rings. For such schemes every connected Y étale over X is irreducible. Normal schemes are geometrically unibranch. See [GD71] IV.6.15.1. for details.

Remark 3.2.23. A general theorem ([AM86] 11.1) shows that the étale homotopy type of a pointed connected geometrically unibranch scheme is profinite, meaning that it is a pro-object of the subcategory of $\mathrm{Ho}(\mathbf{ss}_*)$ where all homotopy groups are finite. See Fact 3.2.29 below.

Interestingly enough, the proof of that reduces quickly to the étale site over the spectrum of a field. By generalised Galois theory we know that this is equivalent to the category of continuous $\mathrm{Gal}(K)$-sets. The general theorem is then proven if all the homotopy groups are finite for a hypercovering of the site of continuous G-sets. We will look at the fundamental group after some more discussions.

Since we are mainly interested in the fundamental group, we will focus on showing that the fundamental pro-group of the homotopy type is profinite. Because we are considering the site of finite G-sets, and since the fundamental group of a simplicial set only relies on 0-, 1- and 2-simplices we can work only with 2-truncated hypercoverings that have finitely many elements.

Example 3.2.24. (*Artin-Mazur Example 9.9*) Let $\mathcal{C}$ be the site of finite left G-sets for a group G with the forgetful functor to sets. Since G is a projective object in this category, in the sense that $\hom(G, -)$ preserves epimorphisms (given a map $G \to X$ and an epi $Y \to X$ we can find a map $G \to Y$ making the triangle commutative) and so

$$K_\bullet = \cdots \quad G \times G \times G \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; G \times G \;\substack{\longrightarrow\\[-2pt]\longrightarrow}\; G$$

dominates all other hypercoverings of this site, by which the pro-object determined by the site is entirely determined by the connected components $\pi(K_\bullet)$. The set of connected components $\pi(K_\bullet)$ is just the orbit space of $K_\bullet$ under the action of G and thus $\pi(K_\bullet) = K(G, 1)$, so that $\pi_1(\mathcal{C}) = G$ and $\pi_q(\mathcal{C}) = 0$ for $q \neq 1$.

Example 3.2.25. (*Artin-Mazur Example 9.11*) If G is a profinite group and $\mathcal{C}$ is the site of finite continuous G-sets, (i.e. a Galois category), then we can consider the n-truncated hypercoverings. From the previous example it follows that they are all dominated by some hypercovering of the form

$$K_\bullet = \cdots \quad \overline{G} \times \overline{G} \times \overline{G} \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; \overline{G} \times \overline{G} \;\substack{\longrightarrow\\[-2pt]\longrightarrow}\; \overline{G}$$

for some finite quotient $\overline{G}$ of G. This gives that $\pi_1(\mathcal{C}) = G$ and that $\pi_q(\mathcal{C}) = 0$ for $q > 1$.

Lemma 3.2.26. *For a group G and a hypercovering $\mathcal{U}_\bullet$ of the site of (finite) G-sets, $\pi(\mathcal{U}_\bullet)$ satisfies the Kan condition in dimension 1.*

Proof. Let G be a group and $\mathcal{U}_\bullet$ a hypercovering. We want to show that $\pi(\mathcal{U}_\bullet)$ satisfies the Kan condition in dimension 1. Let $K_\bullet = \pi(\mathcal{U}_\bullet)$ and let x, y be in K_1 satisfying $d_0 x = d_0 y$. We can choose representatives a, b in $\mathcal{U}_1$ such that $d_0 a = d_0 b$. Theorem 8.4 of [AM86] gives that there is a c in $\mathcal{U}_1$ such that $d_0 c = d_1 a$ and $d_1 c = d_1 b$, i.e. an element (a, b, c) of $(\mathrm{cosk}_1 \mathcal{U}_\bullet)_2$ which as discussed identifies as a subset of the triple intersections of $\mathcal{U}_1$, and that there is an element in $\mathcal{U}_2$ mapping to this element (a, b, c). Taking this element as a representative of w in K_2, we get $d_0 w = x$ and $d_1 w = y$, thus the construction satisfies the Kan condition. $\square$

Since $K_\bullet$ is a finite simplicial set, it follows that the fundamental group is finite since it is a quotient of the finite set K_1, and we have the proposition

Proposition 3.2.27. *The fundamental group $\pi_1(\pi(\mathcal{U}_\bullet))$ of a hypercovering in G-sets is a finite group.*

Since the fundamental pro-group of the étale homotopy type is the system of all the fundamental groups of these hypercovers, i.e. a system of finite groups, we have the following corollary.

Corollary 3.2.28. *The fundamental group of the étale homotopy type of a geometrically unibranch locally Noetherian scheme is profinite.*

Proof. The proof relies on the following observations: when $Y \to X$ is étale and Y is connected, Y is irreducible since X is unibranch. Thus, if Y is connected, $Y \times_X P$ is connected and the connected components are in bijection; $\pi(Y) = \pi(Y \times_X P)$. Here P is the generic point of our scheme X, which we may assume is reduced since the étale site of X and the reduction X_{red} are equivalent. The fibre of P in a connected cover will consist of a single point. Thus we can consider $K_\bullet \times_X P$ instead of $K_\bullet$, and so we have reduced the discussion to the étale site of the spectrum of a field, which we know can be described by G-sets where G is the absolute Galois group of the field. Now the assertion follows from 3.2.27. □

In [AM86] chapter 11, corollary 3.2.28 is used to show that under the assumptions above, all higher homotopy groups of a hypercover $K_\bullet$ is finite, which implies that the étale homotopy type of a geometrically unibranch and locally Noetherian scheme is profinite.

Fact 3.2.29 (Artin-Mazur Theorem 11.1). *For a locally Noetherian and geometrically unibranch scheme X, all homotopy groups $\pi_q(X_{\text{et}})$ are profinite.*

Remark 3.2.30. In Grothendieck's perspective, the topological fundamental group is intrinsically linked with the theory of G-torsors, or *prinicpal G-bundles*, or more precisely, the connected covering spaces corresponding to normal subgroups of $\pi_1(X)$ are principal G-bundles for a quotient group G. A principal G-bundle is a topological space E with a free and continuous action of G such that if $E \to E/G$ is the quotient map, then every point x of E/G lies in some open U whose preimage is homeomorphic to $G \times U$ with the action being $g(h, x) = (gh, x)$. In other words, it is a fibre bundle where the fibres are G with locally trivial actions on them. Common notation is to set $B = E/G$ to the base and write $\pi : E \to B$ or $G \to E \to B$. For instance, any nice connected space X has the universal cover $\tilde{X} \to X$ as a natural principal $\pi_1(X)$-bundle. For a discrete group G, we have a classifying space BG that is connected and has fundamental group G. It is commonly known that BG is an Eilenberg-Maclane space, $K(G, 1)$. For nice spaces we have a theorem:

Theorem 3.2.31. *Let X be connected, homotopy equivalent to a CW-complex and compact and let G be a discrete group. Then there is a chain of bijections*

$$[X, BG] \cong \Pi(X, G) \cong \hom_{\mathbf{Grp}}(\pi_1(X), G)$$

between the homotopy classes of maps $X \to BG$, the principal G-bundles over X and the group homomorphisms $\pi_1(X) \to G$.

Remark 3.2.32. Just like the topological case, the étale fundamental group of chapter 2 is intrinsically linked with the theory of G-torsors. For a finite group G a *G-torsor* (or a *principal G-bundle*) is a scheme Y with a given G-action $G \times Y \to Y$ that is locally trivial in the Grothendieck topology. In [Gro61], G-torsors Y that are finite étale over X are studied and it is shown that $\hom(\pi_1^{\text{et}}(X), G) = [B\pi_1^{\text{et}}(X), BG] = \Pi(X, G)$ where the left-hand side are the continuous group homomorphisms and where the middle is homotopy classes of maps and the right-hand side are the G-torsors. This is the motivation for the following discussion.

Remark 3.2.33. For any site $\mathcal{C}$ we can consider *descent data* for an object $X \in \mathcal{C}$, which is given by a fibred category $\mathcal{J}$ such that for any a, b in $\mathcal{J}$ the functor mapping $f : Y \to X$ to $\hom(f^*a, f^*b)$ is a sheaf on the site $\mathcal{C}/X$. For a hypercover $K_\bullet$ on $\mathcal{C}$ we can consider descent data for $x \in \mathcal{J}(K_0)$ relative $K_\bullet$, given by an isomorphism $\phi : d_0^*x \to d_1^*x$ in $\mathcal{J}(K_1)$ such that in $\mathcal{J}(K_2)$ we have $d_1^*\phi = d_2^*\phi d_0^*\phi$ where the d_i are given by the structure of the simplicial set. Artin and Mazur proves that the map $K_\bullet \to \operatorname{cosk}_0 K_\bullet$ induces an equivalence of descent data for x relative to $K_\bullet$ and relative to $\operatorname{cosk}_0 K_\bullet$. The proof proceeds by writing K_2 as a subobject of $K_1 \times K_1 \times K_1$ of elements (b_0, b_1, b_2) satisfying

$$d_0 b_0 = d_0 b_1$$
$$d_1 b_0 = d_0 b_2$$
$$d_1 b_1 = d_1 b_2$$

which after a diagram chase and using some simplicial identities gives the desired correspondence. This implies that the descent data depends only on the covering $K_0 \to *$ of the terminal object, and on the object $X \in \mathcal{J}(K_0)$.

As a consequence of the general discussions, Artin and Mazur consider the case when $\mathcal{C}$ is closed under arbitrary coproducts and locally connected, e.g. the étale site, with $\mathcal{J}(X)$ the category of objects $X \otimes S$ with S a set, called trivial coverings, with maps being compatible with maps of X. This case is much simpler to grasp: The group of automorphisms of $X \otimes S$ over X is the permutation group $P(S)$ of the set S if X is connected, and $P(S)^{\pi(X)}$ if not.

Thus, if $K_\bullet$ is a hypercovering, the descent data for the trivial covering $K_0 \otimes S$ is given by an automorphism of $K_1 \otimes S$ satisfying the descent conditions, which is the same as an element of $P(S)$ for each connected component of K_1. Thus the category of locally trivial coverings of the terminal object of $\mathcal{C}$ that become trivial on K_0 is equivalent to the category of simplicial covering spaces of $\pi(K_\bullet)$. This is summarised in Corollary 10.6 of [AM86].

Remark 3.2.34. Friedlander considers this in a slightly more general way. He considers the natural definition of *principal G-fibrations* over a simplicial scheme $X_\bullet$ and denotes the isomorphism classes by $\Pi(X_\bullet, G)$. Here a principal G-fibration over $X_\bullet$ is a map $Y_\bullet \to X_\bullet$ with a right action of G on Y over X and an isomorphism $U \times_{X_0} Y_0 \cong U \otimes G$ for an étale surjective map $U \to X_0$. He then proves with similar techniques that $\Pi(X_\bullet, G)$ is equivalent to the combination of $\Pi(X_0, G)$ and descent data. See lemma 5.4 of [Fri82].

A direct consequence of this is the following, which is Corollary 10.7 of [AM86]. It shows that the fundamental pro-group plays the same role for principal G-fibrations over X as the fundamental group of a simplicial set does for the its principal G-fibrations, and the étale fundamental group does for its corresponding G-torsors.

Corollary 3.2.35. *Let $\mathcal{C}$ be a pointed site that is locally connected and closed under arbitrary coproducts. Let $\Pi(\mathcal{C}, G)$ be the isomorphism classes of locally trivial coverings of the terminal object with a given operation by the group G making them principal fibre bundles. Then*

$$\Pi(\mathcal{C}, G) = \hom_{\mathbf{Grp}}(\pi_1(\mathcal{C}), G)$$

In the discussion following the above corollary [AM86] 10.7 Artin and Mazur show that a direct consequence of the corollary is that the fundamental group $\pi_1(X_{\mathrm{et}}, x)$ for a Noetherian scheme X is isomorphic to the enlarged fundamental group from SGA 3.X.6 which we discussed in remark 2.3.6. The profinite completion of the enlarged fundamental group is the usual fundamamental group π_1^{et}, but if the enlarged fundamental group is already profinite (which it is by 3.2.28) they must be isomorphic. Thus for normal schemes (or geometrically unibranch) the three different fundamental groups (the one from the Galois category, the enlarged one, and the one from the étale homotopy type) all agree. Note that if X is also of finite type over $\mathbb{C}$ such that we know its topological fundamental group, we know that its profinite completion is the same as the étale fundamental group and we have four different formalisms giving the same resulting group!

Corollary 3.2.36. *For a connected and geometrically unibranch locally Noetherian scheme X the fundamental group of X_{et} is isomorphic to the étale fundamental group,*

$$\pi_1^{\mathrm{et}}(X) \cong \pi_1(X_{\mathrm{et}}).$$

3.2.c. Friedlander's construction

The Artin-Mazur construction gives an object of pro-Ho($\mathbf{ss}$), but this is not an ideal category for homotopy theory. Friedlander constructed, in [Fri82], a generalisation that gives an object of pro-**ss**, which can then be localised to Ho(pro-**ss**), which is more suited to homotopy theory. Here we will only present some necessary notions for understanding the construction and some immediate consequences, for which chapter 1–5 of the book will be sufficient. The construction is slightly more involved and starts with defining the étale site for a simplicial scheme.

If A is an integral domain, then any Zariski open sets intersect in Spec A, and thus the nerve of a Zariski cover is contractible. However, étale maps need not be inclusions, which makes the étale topology more interesting than the Zariski topology.

Definition 3.2.37. For $X_\bullet$ a simplicial scheme, let $\mathrm{Et}(X_\bullet)$ be the category with objects being étale maps $U \to X_n$ for some integer $n \geq 0$ and maps being commutative squares

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_m \end{array}$$

where the bottom arrow is from the simplicial structure of $X_\bullet$. A cover of $U \to X_n$ is a surjective family $\{U_i \to U\}$ of étale maps over X_n. This gives the *étale site* $\mathrm{Et}(X_\bullet)$. The definition for a bisimplicial scheme is similar, $\mathrm{Et}(X_{\bullet\bullet})$. The reason for using this notation instead of the notation X_{et}, $(X_\bullet)_{\mathrm{Et}}$ et cetera as in Chapter 2 is to avoid confusion with the étale homotopy type X_{et}.

As usual, $\{U_i \to U\}$ can be identified with the étale map $\coprod_i U_i \to U$. For any map $U \to X_n$ we can ask for a simplicial map $U_\bullet \to X_\bullet$ and a relevant result is the following.

Proposition 3.2.38. *For a simplicial scheme $X_\bullet$ and any $n \geq 0$ the map*

$$(-)_n : S\mathbf{Sch}\,/X_\bullet \to \mathbf{Sch}\,/X_n$$

taking $Z_\bullet \to X_\bullet$ to $Z_n \to X_n$ admits a right adjoint Γ_n. Also, if $W \to X_n$ is étale (or surjective) the map $\Gamma_n(W) \to X_\bullet$ is étale (or surjective) in each dimension.

Proof. The construction is rather explicit: we let $\Gamma_n(W)_k$ be the fibre product of X_k and $W^{\times \Delta_n^k}$ over $X_n^{\times \Delta_n^k}$ and then we verify that we get the desired bijection of hom-sets. $\square$

The canonical map $\Gamma_n W \to X_\bullet$ factors through W in dimension n, which means that if $W \to X_n$ is étale or surjective, it is dominated by some $U_\bullet \to X_\bullet$. As expected we define sheaves on our new site:

Definition 3.2.39. A *presheaf* on a simplicial scheme is a functor $P : \mathrm{Et}(X_\bullet)^{\mathrm{op}} \to \mathbf{Set}$ taking $(U \to X_\bullet)$ to the set $P(U)$. A *sheaf* is a presheaf F such that for every cover $\{U_i \to U\}$ the set $F(U)$ is the equaliser of

$$\textstyle\prod_i FU_i \rightrightarrows \prod_{i,j} F(U_i \times_U U_j).$$

In other words, F consists of sheaves F_n on $\mathrm{Et}(X_n)$ and maps $F_n \to a_* F_m$ for a a map in $\mathbf{\Delta}$. In particular, if $X_\bullet = X \times \Delta^0$ then a sheaf on $\mathrm{Et}(X_\bullet)$ is a cosimplicial object of sheaves on Et X. Any scheme W gives a representable sheaf $(U \to X_n) \mapsto \mathbf{Sch}(U, W)$

In the case Spec k for a field k a map $U \to \operatorname{Spec} k$ in $\mathrm{Et}(\operatorname{Spec} k)$ is dominated by a finite Galois Spec $K \to \operatorname{Spec} k$ and a sheaf of abelian groups on $\mathrm{Et}(\operatorname{Spec} k)$ is equivalent to taking the colimit over all such K, which gives the value on $F(\bar{k})$ with a left action of the absolute Galois group so that the stabilisers of any element is a subgroup of finite index. In particular this is equivalent to the value $F(\operatorname{Spec} k)$ if $k = \bar{k}$ is separably closed. This is the usual notion of point.

Definition 3.2.40. A *geometric point* of a simplicial scheme $X_\bullet$ is a map $\operatorname{Spec} k \to X_m$ for some m and some separably closed field k. The *stalk of the sheaf* F at a geometric point $a : \operatorname{Spec} k \to X_m$ is $a^* F_m(\operatorname{Spec} k)$.

Unsurprisingly, the following holds as a consequence of the corresponding fact for ordinary schemes.

Fact 3.2.41. *The category of abelian sheaves on the étale site of a simplicial scheme $X_\bullet$ is an abelian category with enough injectives. A sequence of sheaves is exact if and only if it gives an exact sequence of stalks for any geometric point.*

This allows us to define cohomology functors $H^i(X_\bullet, -)$.

Fact 3.2.42 ([Fri82] 2.5)**.** *For bisimplicial schemes $X_{\bullet\bullet}$ we have a natural isomorphism*

$$H^*(X_{\bullet\bullet}, -) \cong H^*(\Delta X_{\bullet\bullet}, (-)^\Delta)$$

where $\Delta X_{\bullet\bullet}$ is the diagonal of $X_{\bullet\bullet}$ and $(-)^\Delta$ is the restriction of the sheaf from the bisimplicial scheme to its diagonal.

Definition 3.2.43. Precisely as for ordinary schemes, we may define the *Čech nerve* of an étale cover of a simplicial scheme $U_\bullet \to X_\bullet$ as the bisimplicial scheme $N_{X_\bullet}(U_\bullet)_{n,m} = (N_{X_n}(U_n))_m$. Similarly, we define *hypercoverings* of simplicial schemes as bisimplicial schemes $U_{\bullet\bullet}$ such that $U_{n,\bullet} \to X_n$ is a hypercover in the previous sense.

Given an abelian presheaf $P : \operatorname{Et}(X_\bullet)^{\mathrm{op}} \to \mathbf{Ab}$ and an étale map $U_\bullet \to X_\bullet$ we obtain a bi-chain complex $P(N_{X_\bullet}(U_\bullet))$ and we may obtain the cohomology groups from its total complex as usual, and define Čech cohomology

$$\check{H}^i(X_\bullet, P) = \operatorname*{colim}_{U_\bullet \to X_\bullet} H^i(P(N_{X_\bullet}(U_\bullet))).$$

The following definition generalises the behaviour of subsets in a topological space as well as the homotopy category of Čech nerves of coverings $U_\bullet \to X_\bullet$.

The following lemma is left as an exercise by Friedlander, [Fri82].

Lemma 3.2.44. *For a functor $F : \mathcal{C} \to \mathcal{D}$ which gives a map $\operatorname{Ho} F : \operatorname{Ho} \mathcal{C} \to \operatorname{Ho} \mathcal{D}$ where $\operatorname{Ho} \mathcal{C}$ and $\operatorname{Ho} \mathcal{D}$ are obtained by imposing equivalence relations on the hom-sets, if $\operatorname{Ho} \mathcal{C}$ and $\operatorname{Ho} \mathcal{D}$ are left filtering and F admits a right adjoint, then $\operatorname{Ho} F$ is left final.*

Proof. Let G be the right adjoint of F and recall that $\hom(Fc, d) \cong \hom(c, Gd)$ is a natural isomorphism. For the first condition of 1.3.11, we want to show that for all d there is a c such that $Fc \to d$, and this is evident by sending the identity map $Gd \to Gd$ to $FGd \to d$ in our bijection, where $Gd = c$ is the desired element. For the second condition, we assume that we have two maps $Fc \to d$ in $\operatorname{Ho} \mathcal{D}$. We can send representatives of these to $c \to Gd$. Since $\operatorname{Ho} \mathcal{C}$ is left filtering, there is some $c' \to c$ making the compositions commute in $\operatorname{Ho} \mathcal{C}$, and sending the map back through the adjoint bijection gives $Fc' \to Fc$ such that the maps $Fc' \to \ Fc \rightrightarrows d$ are equal in $\operatorname{Ho}(\mathcal{D})$. □

The lemma 3.2.44 gives the following propostion:

Proposition 3.2.45 ([Fri82] 3.4)**.** *For a simplicial scheme $X_\bullet$ the homotopy category of hypercoverings $HR(X_\bullet)$ is left filtering and the restriction $HR(X_\bullet) \to HR(X_n)$ is left final for all $n \geq 0$.*

Remark 3.2.46. Returning once again to remarks about cohomology, we can note that a further generalisation of the Verdier hypercovering theorem is proven in [Fri82] as Theorem 3.8, and as a consequence we know that the sheaf cohomology is isomorphic to the Čech cohomology for many cases of interest, or in other words, we have the following theorem.

Theorem 3.2.47.. *For a simplicial scheme $X_\bullet$ there is a natural isomorphism*

$$H^*(X_\bullet, -) \to \operatorname*{colim}_{U_{\bullet\bullet} \in HR(X_\bullet)} H^*((U_{\bullet\bullet})).$$

Corollary 3.2.48.. *From the inclusion of Čech nerves into hypercovers we see that if all X_n is quasi-projective over a Noetherian ring, then we have an isomorphism $\check{H}^*(X_\bullet, -) \to H^*(X_\bullet, -)$.*

3.2.d. The étale topological type

We are now almost ready to describe the construction of the topological type, but we need some more technical details first. To make the coverings better behaved we give every connected component a geometric basepoint.

Fact 3.2.49. *Recall that if a scheme X is locally Noetherian then its connected components are open and closed.*

Definition 3.2.50. The set of geometric points of X is denoted $\overline{X}$. A *rigid covering* $a : U \to X$ for a locally Noetherian X is a disjoint union of pointed separated étale maps $a_x : (U_x, u_x) \to (X, x)$, one for every $x \in \overline{X}$, such that all U_x are connected and $a_x u_x = x$. Maps of rigid coverings $(a : U \to X)$ to $(b : V \to Y)$ are defined as squares

$$\begin{tikzcd} U \arrow[r] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r] & Y. \end{tikzcd}$$

Proposition 2.1.59 implies that two rigid covers has at most one map between them. Thus the category $RC(X_\bullet)$ of rigid coverings is left directed. Similarly, we may define $HRR(X_\bullet)$, the category of rigid hypercoverings where the maps

$$U_{s,t} \to (\operatorname{cosk}_{t-1}^{X_s} U_{s,\bullet})_t$$

are rigid coverings with compatible maps for the simplicial structure maps

$$(\operatorname{cosk}_{t-1}^{X_s} U_{s,\bullet})_t \to (\operatorname{cosk}_{t-1}^{X_{s'}} U_{s',\bullet})_t$$

corresponding to $[s] \to [s']$. The maps of rigid hypercovers over a map $f : X \to Y$ is a map $\phi : U_{\bullet\bullet} \to V_{\bullet\bullet}$ such that $U_{s,t} \to V_{s,t}$ is a map of rigid covers over $\operatorname{cosk}_{t-1} \phi$ for all s and t. The $t-1$-truncation of a map $\phi_{s,\bullet}$ determines the map $\operatorname{cosk}_{t-1} \phi_{s,\bullet}$ completely, and by this it follows that there is at most one map between two rigid hypercoverings from the corresponding fact for the rigid coverings it consists of. By constructing a rigid product $U_{\bullet\bullet} \times^R_{X_\bullet} V_{\bullet\bullet} \to X_\bullet$ from two rigid hypercoverings $U_{\bullet\bullet}$ and $V_{\bullet\bullet}$, Friedlander shows that this category is a left filtering category in his proposition 4.3.

Definition 3.2.51. A *strict map* of pro-objects $F : I \to \mathcal{C}$ and $G : J \to \mathcal{D}$ is a functor $a : J \to I$ and a transformation Fa to G. Of course, the identity transformation gives a strict map from F to Fa for every functor a.

If $a : J \to I$ is left-final then there exists a natural transformation $Fa \to F$ which is an inverse in the sense that Fa and F are isomorphic as pro-objects.

Construction 3.2.52. The notation $f^* : HRR(Y_\bullet) \to HRR(X_\bullet)$ means the following. For $V \to Y$ a rigid covering and a map $f : X \to Y$ we construct a *rigid pullback*. As we have a collection of maps $(V_y, v_y) \to (Y, y)$ for every point y of Y, we consider $V_{fx} \times_Y X \to X$ for x of X. Let

$(V_{fx} \times_Y X)_o$ denote the connected component containing the geometric point $fx \times x$, and then the rigid pullback is the disjoint union of these connected components,

$$\begin{array}{ccc} \coprod (V_{fx} \times_Y X)_o & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y. \end{array}$$

Definition 3.2.53. The *étale topological type* $(X_\bullet)_{\text{et}}$ of a simplicial scheme $X_\bullet$ is the functor $\pi\Delta$: $HRR(X_\bullet) \to$ **ss** sending a rigid hypercovering $U_{\bullet\bullet} \to X_\bullet$ to the simplicial set $\pi(\Delta U_{\bullet\bullet})$ of connected components of the diagonal $(\Delta U_{\bullet,\bullet})_n = U_{n,n}$. A map $f : X_\bullet \to Y_\bullet$ of locally Noetherian simplicial schemes gives maps $f^* V_{\bullet\bullet} \to V_{\bullet\bullet}$ and thus induces transformations $(X_\bullet)_{\text{et}} \circ f^* \to (Y_\bullet)_{\text{et}}$. This gives a map f_{et} of pro-simplicial sets.

Remark 3.2.54. Proposition 4.5 of [Fri82] shows that if we consider a scheme X and we consider the simplicial scheme $X = (X \otimes \Delta[0])$, mapping the pro-simplicial set X_{et} forgetfully to the homotopy category pro-$\mathcal{H}$ of CW-complexes, then this is isomorphic to Artin and Mazurs homotopy type. Thus this whole expedition into bisimplicial schemes fulfills its sole purpose of giving an object of pro-**ss** instead of pro-$\mathcal{H}$. This is because the category pro-$\mathcal{H}$ is not a suitable category for homotopy theory. The category of pro-simplicial sets however, can be given several natural model structures. In section 4.2.1 we will briefly look at one approach by Isaksen from [Isa01] to define such a model structure, and indicate what benefits it brings.

There are varying conventions for the notation regarding the étale homotopy type. [AM86] originally used $\text{Et}(X)$ for the étale homotopy type of X (in pro-Ho(**ss**)), whereas [Fri82] uses X_{et} for the topological type (in pro-**ss**) and $\text{Et}(X)$ for the étale site. In modern usage the term *étale homotopy type* is sometimes denoted by X_{ht} and usually refers to an object of Ho(pro-**ss**) rather than pro-Ho(**ss**) but since the Friedlander definition is a true generalisation of the Artin-Mazur definition, the various nuances between these notations is sometimes neglected through various forms of abusive notation.

Chapter 4

The anabelian conjectures

4.1. Anabelian geometry

4.1.a. Motivation and historical overview

This exposition owes a lot to several different sources, notably the book by Szamuely, [Sza09] and an oral presentation given by Alexander Schmidt in Munich, [Sch17] and other expository articles on the subject.

In 1983 Grothendieck wrote a letter to Faltings [Gro83], which was concerned with so-called *anabelian* (in the original German: *anabelsch*) schemes over a finitely generated extension of $\mathbb{Q}$. Although he does not give a definition of what an anabelian scheme X is apart from that it should be reconstructible from its étale fundamental group, Grothendieck conjectures that anabelian schemes do in fact exist. The various conjectures in the letter have been summarised by Alexander Schmidt as follows:

Conjectures 4.1.1. *The conjectures:*

1. *If C is a smooth hyperbolic curve (i.e. one with negative Euler characteristic) then C is anabelian.*
2. *If X is a smooth scheme, then every point of X has a fundamental system of Zariski neighbourhoods that are anabelian.*
3. *If X is anabelian the rational points $X(k)$ (= sections of the structure map $X \to \operatorname{Spec} k$) are reconstructible from the Galois sections of $\pi_1(X) \to G_k$.*

Here G_k is the absolute Galois group of k. The third conjecture is sometimes referred to as *the section conjecture*. We will otherwise refer to them as conjecture 1, 2 and 3. One fundamental issue in interpreting these conjectures, is to say what the word *reconstructible* means. Here are three possible meanings for X, Y smooth:

- ("weak") If $\pi_1(X) \cong \pi_1(Y)$ then $X \cong_k Y$.
- ("strong") The natural map $\operatorname{Isom}_k(Y, X) \to \operatorname{Isom}_{G_k}(\pi_1(Y), \pi_1(X))/\pi_1(X_{\overline{k}})$ is an isomorphism.
- ("Yoneda") For every Y the map $\hom_k^{\mathrm{dom}}(Y, X) \to \hom_{G_k}^{\mathrm{open}}(\pi_1(Y), \pi_1(X))/\pi_1(X_{\overline{k}})$ is an isomorphism.

One of the first results that could be considered "anabelian" even though it predates the word, is the Neukirch-Ikeda-Uchida theorem, which states that the isomorphism classes of number fields are determined by the isomorphism classes of their absolute Galois groups.

Theorem 4.1.2 (Neukirch-Ikeda-Uchida). *Let K and L be two number fields. If $\operatorname{Gal}(K)$ and $\operatorname{Gal}(L)$ are isomorphic as topological groups, then K and L are isomorphic as fields.*

The work by Uchida was to show that an automorphism of the number field correspond bijectively to an outer automorphism of the absolute Galois group. In other words, we can reconstruct the field from its Galois group. In the Grothendieck perspective, the Galois group is indeed the fundamental group of the spectrum of the field, with the chosen algebraic closure as basepoint. The formalism of Galois categories, as discussed in 2.3.1, proved beneficial. Grothendieck showed many important results for his new fundamental group, for instance that given an open set U of an integral proper normal curve X over a field k, the genus g of $X_{\overline{k}}$ and the number n of closed points in $X - U$ for an open set U entirely determines the maximal prime-to-p quotient of $\pi_1(U)$, where p is the characteristic of the field k. In characteristic 0, it follows by comparison to the complex case that the fundamental group is entirely determined by the genus.

In the letter to Faltings, Grothendieck claims that the outer Galois representation

$$\operatorname{Gal}(\overline{k}|k) \to \operatorname{Out}(\pi_1(U_{\overline{k}}))$$

should determine $\pi_1(U)$ in certain cases, in particular when an integral normal curve U is a *hyperbolic curve*, one for which $2g-2+n$ as above is greater than zero, or equivalently a curve of negative Euler characteristic. One of Falting's students, Mochizuki, set out to work on these new anabelian conjectures. He and Tamagawa proved a couple of important results that built on each other, with Tamagawa proving the affine case and Mochizuki extending it. Mochizuki then used a completely different approach of p-adic Hodge theory, as developed by Faltings, to prove the following.

Theorem 4.1.3 (Mochizuki)**.** *A hyperbolic curve U over a sub-p-adic field K (a subfield of a finitely generated extension of $\mathbb{Q}_p$) is determined up to isomorphism by the outer Galois representation.*

The proof of this theorem (and the next one) is published in [Moc99]. The (continuous) maps of profinite groups $G_1 \to G_2$ over G, with $G_i \to G$ surjective make up the set $\hom_G(G_1, G_2)$, and this set gets a left action by the inner automorphisms of G_2 and the quotient is defined as $\hom_G^{\mathrm{Out}}(G_1, G_2)$. The case in Mochizuki's theorem is when $G = \mathrm{Gal}(K)$ and $G_i = \pi_1^{(p)}(U_i)$. If a map $\phi : U_1 \to U_2$ is dominating, the map of fundamental groups has open image, and the correspondence can be written as

Theorem 4.1.4 (Mochizuki)**.** *For two hyperbolic curves U_1, U_2 over a sub-p-adic field K, we have a bijection*

$$\hom_K^{\mathrm{dom}}(U_1, U_2) \to \hom_{\mathrm{Gal}(K)}^{\mathrm{Out,open}}(\pi_1^{(p)}(U_1), \pi_1^{(p)}(U_2)).$$

Thus, in a sense, the curves can be reconstructed to a certain degree from their fundamental groups. If we consider the exact sequence

$$1 \to \pi_1(X_{\bar{k}}) \to \pi_1(X) \to \mathrm{Gal}(\bar{k}|k) \to 1$$

for X a proper integral normal curve over k, and let x be a k-rational point, we can let $\tilde{x}$ be a pro-point on the profinite cover $\tilde{X} \to X$ as in remark 1.4.18. The stabiliser of $\tilde{x}$ in $\pi_1(X)$ is taken to a subgroup of $\mathrm{Gal}(k)$ and is in fact isomorphic to $\mathrm{Gal}(k)$. This gives a section of the map $\pi_1(X) \to \mathrm{Gal}(k)$ and a conjugacy class of sections by the action of $\pi_1(X)$ corresponding to the choice of pro-point $\tilde{x}$. In other words there is a map ϕ taking a k-rational point to a section. A variant of the section conjecture states the following.

Conjecture 4.1.5. *If X is a curve of genus $g \geq 2$ over a field k that is finitely generated over $\mathbb{Q}$, the map*

$$\phi : X(k) \to \hom_{\mathrm{Gal}(k)}^{\mathrm{Out}}(\mathrm{Gal}(k), \pi_1(X))$$

is a bijection, or in other words all conjugacy classes of sections come from a unique rational point of X.

Grothendieck knew how to prove the injectivety, but the surjectivety remains open.

Remark 4.1.6. Let us assume that we have a higher-dimensional scheme. Unless the scheme is an étale $K(\pi, 1)$-space, it seems implausible that it should be uniquely determined by its fundamental group alone as there are higher homotopy groups that can be taken into account. It makes sense to instead ask the more general question of schemes reconstructible from their homotopy type rather than just the first homotopy group.

4.2. Pro-geometry preliminaries

4.2.a. Pro-geometry and Isaksen's model structure

Recall from section 1.3.1 that for any category $\mathcal{C}$ the pro-category pro-$\mathcal{C}$ is defined with

- Objects: functors $X : I^{\mathrm{op}} \to \mathcal{C}$ where I is small and filtering.
- Morphisms: $\hom_{\text{pro-}\mathcal{C}}((X_i),(Y_j)) = \lim_i \operatorname{colim}_j \hom(X_i, Y_j)$.

For the category of simplicial sets (sometimes called *spaces*) **ss** we have a pro-category pro-**ss** which has a pointed version pro-$\mathbf{ss}_*$. For a pro-space $X = (X_i) \in$ pro-**ss** we can use the homotopy group functors π_n level-wise and obtain a pro-group $\pi_n(X) = (\pi_n(X_i))$.

In [Isa01], Isaksen defines a model structure on the category of (pointed) pro-simplicial sets, which enables us to do homotopy theory with the étale homotopy type $X_{\mathrm{et}} \in \mathrm{Ho}(\text{pro-}\mathbf{ss}_*)$ of a scheme X. Informally, the model structure on pro-**ss** is defined from the standard model structure on **ss** by

- Cofibrations: (isomorphic to) level-wise cofibrations.
- Weak equivalences: "isomorphisms on homotopy groups".

This latter statement is formalised with the notion of local systems (roughly, viewing the homotopy groups as locally constant sheaves) and the class of fibrations is obtained by the model category requirements from the other two classes.

Remark 4.2.1. "Pro-spaces are beasts", [Sch17]:

- They do not always have points. A point is a morphism $* \to X$. The space $X_n = ([n,\infty))$ has no points, as any image of a constant function will be outside the interval $[n,\infty)$ for large enough n.
- The homotopy groups $\pi_n(X,x)$ may depend on the basepoint $x \in X$ even if X is connected. Thus questions of the type "unpointed versus pointed" are subtle in Ho(pro-**ss**)!

To better handle the problems of pointed and unpointed fundamental groups, we define *topological fundamental groups*. This is in section A.2.2 of [SS16].

Definition 4.2.2. For $X \in$ pro-$\mathbf{ss}_*$ we define $\pi_n^{\mathrm{top}}(X,x) = [S^n, X]_* = \hom_{\mathrm{Ho}(\text{pro-}\mathbf{ss}_*)}(S^n, X)$, the *topological fundamental group* of (X,x). If $\pi_0^{\mathrm{top}}(X,x) = 0$ we say that the pro-space is *path-connected*.

Using the Bousfield-Kan spectral sequence and a lemma of derived inverse limits (lemma A7 in [SS16]) Stix and Schmidt shows the following proposition.

Proposition 4.2.3 ([SS16] A8)**.** *Let (X,x) be a path-connected pro-space such that all its homotopy groups $\pi_n(X,x)$ are pro-finite for $n \geq 1$. Then the natural map*

$$\pi_n^{\mathrm{top}}(X,x) \to \lim \pi_n(X_i, x_i)$$

is an isomorphism for all $n \geq 0$.

Note that this applies to path-connected étale homotopy types of normal locally Noetherian schemes, since their homotopy groups are pro-finite by 3.2.29.

4.2.b. Covering theory and classifying spaces

Consider the case when X is a CW-complex and H a subgroup of $\pi_1(X)$. The corresponding covering $X_H \to X$ is characterised by the property that for every W the covering identifies $[W, X_H]$ with the maps in $[W, X]$ that map $\pi_1(W)$ into H.

For X in pro-$\mathcal{H}$, and a sub-pro-group H of $\pi_1(X)$ we can assume that H and X are filtered by the same category and have compatible maps $H_j \to \pi_1(X_j)$. The above characterisation applied levelwise gives spaces for every H_j and X_j and thus pro-maps $X_H \to X$. For a $W \to X_j$ to factorise through $X_{j_{H_j}}$ is equivalent to $\pi_1(W) \to \pi_1(X_j)$ factorising through H_j. This gives the following:

Proposition 4.2.4 ([AM86] 2.8)**.** *Since $[W, X_H]_* = \varprojlim_j [W, X_{j_{H_j}}]_*$ we get that for any X in* pro-$\mathcal{H}_*$ *with $H \to \pi_1(X)$ a monomorphism of pro-groups, there is a unique X_H with a map to X such that it induces a factorisation through H of $\pi_1(W) \to \pi_1(X)$.*

This is notably independent of the choices in the prior discussion. In an analogous way to how this is shown from the covering theory of $\mathcal{H}$, the covering theory of **ss** gives a result for pro-**ss**.

Proposition 4.2.5 ([SS16] A2)**.** *For (X, x) in* $\mathrm{Ho}(\text{pro-}\mathbf{ss}_*)_c$ *(where c denotes the subcategory of connected pro-spaces) with a monomorphism $h : H \to \pi_1(X, x)$ there exists a unique $(X, x)_H$ such that $[W, X_H]_*$ is in bijection to the set of maps f in $[W, X]_*$ where $\pi_1(f)$ factors through H by the mapping $f' \mapsto f = hf'$.*

Remark 4.2.6. (*Classifying spaces*) Consider a group G as a one-object category and let BG be its nerve. It is a connected and pointed space with the property that

$$\pi_n(BG) = \begin{cases} G \text{ for } n = 1 \\ 0 \text{ for } n \geq 2 \end{cases}$$

or in other words, a good model for a $K(G, 1)$-space which is functorial. Whenever $G \to G'$ is a surjective homomorphism, the induced map $BG \to BG'$ is a fibration in $\mathbf{ss}_*$, and by setting $G' = *$ we see that BG is a fibrant object. The functorial properties allows us to extend it naturally to the pro-category and gives BG as a pointed pro-space for a pro-group G, which however is not fibrant.

To get a fibrant replacement, we can consider EG, defined as the nerve of the category with object set G and $\hom(a, b) = \mathrm{Isom}(a, b) = *$ for all a, b in G. For any pro-group $G = (G_i)$ we have a pro-group $G^* = (\prod_{j \leq i} G_i)$ which in turn gives $B^*G = EG^*/G$, which is a fibrant object which has a natural weak equivalence $BG \to B^*G$ by Lemma A14 in [SS16].

Using this fibrant replacement, lemma A15 verifies that the commuting triangle of natural maps

$$B : \hom_{\text{pro-}\mathbf{Grp}}(G', G) \to \hom_{\text{pro-}\mathbf{ss}}(BG', BG)$$

$$\hom_{\text{pro-}\mathbf{ss}}(BG', BG) \to \hom_{\mathrm{Ho}(\text{pro-}\mathbf{ss})}(BG', BG)$$

$$\pi_1 : \hom_{\mathrm{Ho}(\text{pro-}\mathbf{ss})}(BG', BG) \to \hom_{\text{pro-}\mathbf{Grp}}(G', G)$$

are all isomorphisms.

Proposition 4.2.7 ([SS16] A16)**.** *For a connected pointed pro-space X and a pro-group G we have a functorial isomorphism*

$$\pi_1 : \hom_{\mathrm{Ho}(\text{pro-}\mathbf{ss}_*)}(X, BG) \to \hom_{\text{pro-}\mathbf{Grp}}(\pi_1(X, x), G).$$

Proof. Let X be in **ss**. For any point x in X the fundamental group $\pi_1(X, x)$ can be considered as a full subcategory of the fundamental groupoid ΠX with the single object x. This gives a functorial map $\pi_1(X, x) \to \Pi X$ to which we can apply the nerve functor and get a map $B\pi_1(X, x) \to B\Pi X$. We can let $B\Pi X$ be pointed by the point x, and note that the map $B\pi_1(X, x) \to (B\Pi, x)$ is a weak equivalence if X is connected. There is also a natural map $X \to B\Pi X$.

The functoriality also allows us to extend the map $B\pi_1(X, x) \to B\Pi X$ to a map of pro-spaces. Let now with rather abusive notation X denote a pro-space instead of a space. If X is a connected pro-space the map $B\pi_1(X, x) \to (B\Pi X, x)$ trivially is a level-wise weak equivalence of simplicial sets, and thus a weak equivalence of pro-spaces, meaning that it has an inverse

$$i : (B\Pi X, x) \to B\pi_1(X, x)$$

in the homotopy category. Composing with the natural arrow $(X,x) \to (B\Pi X, x)$ we get a map p_X in $\mathrm{Ho}(\text{pro-}\mathbf{ss}_*)$

$$\begin{tikzcd} X \arrow[d] \arrow[dd, bend right=60, "p_X"'] \\ B\Pi X \arrow[d, shift left, "\wr"] \\ B\pi_1(X,x) \arrow[u, shift left] \end{tikzcd}$$

In the special case for $(X,x) = (BG,*)$ we have $B\pi_1(BG,*) \to (B\Pi BG, *)$

$$\begin{tikzcd} BG \arrow[d] \arrow[dd, bend right=60, "p_{BG}"'] \\ B\Pi BG \arrow[d, shift left, "\wr"] \\ B\pi_1(BG,*) \arrow[u, shift left] \end{tikzcd}$$

which yields the identity $BG \to B\pi_1(BG,*) = BG$, giving that the map p_{BG} is the identity map.

Now to the proposition: Let $f : (X,x) \to BG$ be a map in $\mathrm{Ho}(\text{pro-}\mathbf{ss}_*)$. Applying $B\pi_1$ and identifying $B\pi_1 BG = BG$, this gives a map $B\pi_1 f : B\pi_1(X,x) \to BG$. We observe that by naturality of the construction we have

$$\begin{tikzcd} (X,x) \arrow[r, "f"] \arrow[d, "p_X"] & BG \arrow[d, "p_{BG}=1_{BG}"] \\ B\pi_1(X,x) \arrow[r, "B\pi_1 f"'] & B\pi_1(BG) = BG. \end{tikzcd}$$

This shows that any f can be written $f = B\pi_1 f \circ p_X$, or in other words we have a surjection

$$p_X^* : \hom_{\mathrm{Ho}(\text{pro-}\mathbf{ss}_*)}(B\pi_1(X,x), BG) \to \hom_{\mathrm{Ho}(\text{pro-}\mathbf{ss}_*)}((X,x), BG).$$

However, the set on the left is in bijection with $\hom_{\text{pro-}\mathbf{Grp}}(\pi_1(X,x), G)$ by the remark 4.2.6 on classifying spaces, and we get a commutative triangle

$$\begin{tikzcd} \hom_{\mathrm{Ho}(\text{pro-}\mathbf{ss}_*)}(B\pi_1(X,x), BG) \arrow[r, two heads, "p^*"] \arrow[d, "\cong"] & \hom_{\mathrm{Ho}(\text{pro-}\mathbf{ss}_*)}((X,x), BG) \arrow[dl, "\pi_1"] \\ \hom_{\text{pro-}\mathbf{Grp}}(\pi_1(X,x), G) & \end{tikzcd}$$

and as the map $X \to B\Pi X$ is surjective, the top map is also injective and thus all arrows in the diagram are bijective, proving the claim. $\square$

The above can be summarised in the following proposition:

Proposition 4.2.8. *Let G be a pro-group. There exists a canonical classifying space $BG \in \mathrm{Ho}(\text{pro-}\mathbf{ss})$ with the usual property $[X, BG]_* \cong \hom_{\text{pro-}\mathbf{Grp}}(\pi_1(X), G)$ for all pointed and connected pro-spaces X.*

This allows us to define what we mean by étale $K(\pi,1)$-spaces.

Definition 4.2.9. A connected $X \in \text{pro-}\mathbf{ss}_*$ is of type $K(\pi,1)$ if and only if $X \to B\pi_1(X)$ is a weak equivalence, which is if and only if $\pi_n(X) = 0$ for all $n \geq 2$.

4.3. The first main result

4.3.a. Reformulating the Mochizuki theorem

Let us now turn back to the geometric side of the discussion. As we have noted, spaces of type $K(\pi, 1)$ are very important in the discussion, and deserves some extra attention.

Example 4.3.1. Schmidt and Stix recalls some well-known examples of $K(\pi, 1)$-varieties.

1. If C is a connected smooth curve which is affine or has genus $g > 0$ over a field k, then it is of type $K(\pi_1(C), 1)$.
2. A finite product of geometrically connected and geometrically unibranch $K(\pi, 1)$ varieties over a field of characteristic zero is also a $K(\pi, 1)$-variety.

This is 2.7 of [SS16].

The discussion of orbits with respect to the topological fundamental group of definition 4.2.2 in the discussions of the pointed versus unpointed case for pro-spaces in section A.2.3 and A.2.4 of the appendix of [SS16] gives the following result as an easy consequence.

Proposition 4.3.2 ([SS16] 2.4)**.** *For X and Y connected varieties over a field k such that Y is geometrically unibranch and geometrically connected. Let K be a sebarably closed extension of k that gives geometric points $\overline{x}, \overline{y}$. Let $\overline{k}$ be the separable closure of k in K.*

1. *Forgetting the base point yields a surjection*

$$\hom_{\mathrm{Ho}(\text{pro-}\mathbf{ss}_*)/k_{\mathrm{et}}}((X_{\mathrm{et}}, \overline{x}_{\mathrm{et}}), (Y_{\mathrm{et}}, \overline{y}_{\mathrm{et}})) \to \hom_{\mathrm{Ho}(\text{pro-}\mathbf{ss})/k_{\mathrm{et}}}(X_{\mathrm{et}}, Y_{\mathrm{et}}).$$

2. *The map in 1. factors through the orbit space for the action of $\pi_1^{\mathrm{et}}(Y_{\overline{k}}, \overline{y})$ and induces a surjection*

$$(\hom_{\mathrm{Ho}(\text{pro-}\mathbf{ss}_*)/k_{\mathrm{et}}}((X_{\mathrm{et}}, \overline{x}_{\mathrm{et}}), (Y_{\mathrm{et}}, \overline{y}_{\mathrm{et}})))_{\pi_1^{\mathrm{et}}(Y_{\overline{k}}, \overline{y})} \to \hom_{\mathrm{Ho}(\text{pro-}\mathbf{ss})/k_{\mathrm{et}}}(X_{\mathrm{et}}, Y_{\mathrm{et}}).$$

3. *If the Galois group G_k is strongly centre-free, the map in 2. is a bijection.*

It is worth noting that sub-p-adic fields were shown to have strongly centre-free absolute Galois groups by Mochizuki in [Moc99], which is what makes this assumption interesting. Combining 4.2.7 and 4.3.2 gives the following lemma.

Lemma 4.3.3 ([SS16] 2.9)**.** *For varieties X, Y, Z over a field k such that $\mathrm{Gal}(k)$ is strongly centre-free, and such that Y, Z are geometrically connected and geometrically unibranch and of type $K(\pi, 1)$, the natural map*

$$\hom_{\mathrm{Ho}(\text{pro-}\mathbf{ss})/k_{\mathrm{et}}}(X_{\mathrm{et}}, (Y \times_k Z)_{\mathrm{et}}) \to \hom_{\mathrm{Ho}(\text{pro-}\mathbf{ss})/k_{\mathrm{et}}}(X_{\mathrm{et}}, Y_{\mathrm{et}}) \times \hom_{\mathrm{Ho}(\text{pro-}\mathbf{ss})/k_{\mathrm{et}}}(X_{\mathrm{et}}, Z_{\mathrm{et}})$$

is a bijection.

Proof sketch. By letting $W = Y \times_k Z$, which is of type $K(\pi, 1)$, proposition 4.2.7 implies that the corresponding pointed version holds for $(X_{\mathrm{et}}, \overline{x}_{\mathrm{et}})$, $(k_{\mathrm{et}}, \overline{k}_{\mathrm{et}})$ and similarly considering Y, Z, W and the orbit sets as in 4.3.2 gives the result. □

Recall that Mochizuki proved the following theorem.

Theorem 4.3.4 ([Moc99] Thm A)**.** *Let X be a smooth connected k-variety and Y a smooth hyperbolic curve over k, where k is a sub-p-adic field. For any choice of geometric base points the natural map*

$$\hom_k^{\mathrm{dom}}(X, Y) \to \hom_{G_k}^{\mathrm{open}}(\pi_1^{\text{ét}}(X, \overline{x}), \pi_1^{\text{ét}}(Y, \overline{y}))_{\pi_1^{\text{ét}}(Y_{\overline{k}}, \overline{y})}$$

is a bijection.

For a homotopy theoretical consideration we consider an algebraic closure $\bar{k}$ of k and view $\bar{k}_{\text{et}}$ as a base point of k_{et}. We can choose base points $\bar{x} \in X(\bar{k})$ and $\bar{y} \in Y(\bar{k})$ that are compatible with $\bar{k}_{\text{et}}$.

$$\begin{tikzcd}
\hom_k^{\text{dom}}(X,Y) \arrow[r] \arrow[d] & \hom_{G_k}^{\text{open}}(\pi_1^{\text{et}}(X,\bar{x}),\pi_1^{\text{et}}(Y,\bar{y}))_{\pi_1^{\text{et}}(Y_{\bar{k}},\bar{y})} \\
\hom^{\pi_1-\text{open}}_{\text{Ho}(\text{pro-}\mathbf{ss})/k_{\text{et}}}(X_{\text{et}},Y_{\text{et}}) & \hom^{\pi_1-\text{open}}_{\text{Ho}(\text{pro-}\mathbf{ss}_*)/(k_{\text{et}},\bar{k}_{\text{et}})}((X_{\text{et}},\bar{x}_{\text{et}}),(Y_{\text{et}},\bar{y}_{\text{et}}))_{\pi_1^{\text{et}}(Y_{\bar{k}},\bar{y})} \arrow[l] \arrow[u, "\pi_1"]
\end{tikzcd}$$

The top arrow is a bijection by Mochizukis theorem. The arrow on the right is a bijection because hyperbolic curves are of type $K(\pi, 1)$ by 4.3.1 which allow us to apply 4.2.7. The bottom arrow is the bijection in 4.3.2. Thus the left arrow is also a bijection and we may conclude the following:

Proposition 4.3.5 ([SS16] 3.2)**.** *With notations as above, the natural map*

$$\hom_k^{\text{dom}}(X,Y) \to \hom^{\pi_1-\text{open}}_{\text{Ho}(\text{pro-}\mathbf{ss})/k_{\text{et}}}(X_{\text{et}},Y_{\text{et}})$$

is a bijection.

Definition 4.3.6. Let Y be a locally closed subscheme in a product of smooth geometrically connected curves C_i over k, $j : Y \to W = C_1 \times ... \times C_n$ with projections $p_i : W \to C_i$. We say that the map j is *factor dominant* if $p_i j$ is dominant for all $i = 1, ..., n$.

When Y is geometrically connected and geometrically reduced over k the compositions $p_i j$ are either dominant or constant and one may remove those where it is constant to obtain an immersion without loss of generality.

Proposition 4.3.7 ([SS16] 4.3)**.** *Let k be a sub-p-adic field and let $j : Y \to W$ be a factor dominant immersion of a geometrically connected and geometrically unibranch variety Y over k into a product $W = C_1 \times ... \times C_n$ of hyperbolic curves as above. Let X be a smooth and connected variety over k. Then any π_1-open morphism $\gamma : X_{\text{et}} \to Y_{\text{et}}$ in* $\text{Ho}(\text{pro-}\mathbf{ss}/k_{\text{et}})$ *has a unique map of k-varieties $f : X \to W$ such that*

$$\begin{tikzcd}
X_{\text{et}} \arrow[d, "\gamma"] \arrow[dr, "f_{\text{et}}"] & \\
Y_{\text{et}} \arrow[r, "j_{\text{et}}"] & W_{\text{et}}
\end{tikzcd}$$

commutes.

Proof. For $n = 0$, $Y = \operatorname{Spec} k = W$ and $f : X \to \operatorname{Spec} k$ satisfies the conditions. For $n > 0$ by 4.3.5 we have unique maps $f_i : X \to C_i$ such that $(f_i)_{\text{et}} = (p_i j)_{\text{et}}\gamma$. Collecting these maps for different i and applying 4.3.3 we obtain a unique map $f : X \to W$ satisfying the conditions. □

The following rather technical result is needed for the main theorem. The proof relies on proposition 4.1 of [SS16] which provides a trace formula for counting points in the closed fibres by l-adic étale cohomology, Poincaré duality and the Leray spectral sequence.

Lemma 4.3.8 ([SS16] 4.6)**.** *Let S be a regular connected scheme of finite type over $\mathbb{Z}$ and let η be its generic point. Let $C_i \to S$ with $i = 1, ..., n$ be proper smooth relative curves whose fibres are connected and of strictly positive genus. Let $W = C_1 \times_S \cdots \times_S C_n$ and let $j : Y \to W$ be a locally closed subscheme which is smooth in* **Sch** */S. For X smooth over S and an S-morphism $f : X \to W$, assume that $X \to S \leftarrow Y$ have "nice" relative compactifications mapping to W. If there is a homotopy equivalence, an isomorphism in* $\text{Ho}(\text{pro-}\mathbf{ss})$

$$\gamma : (X_\eta)_{\text{et}} \to (Y_\eta)_{\text{et}}$$

with $(f_\eta)_{\text{et}} = (i_\eta)_{\text{et}}\gamma$ then f factors uniquely through j,

$$\begin{tikzcd} X \arrow[d, dashed, "g"'] \arrow[dr, "f"] & \\ Y \arrow[r, "j"] & W \end{tikzcd}$$

Here "nice" relative compactification is meant in a technical sense which allows the usage of [SS16] 4.1 to prove the next proposition. (More precisely $\overline{f} : \overline{X} \to S$ should be a proper smooth equidimensional morphism of relative dimension, with $X \subset \overline{X}$ an open complement of a strict normal crossing divisor $D = \bigcup_{i=1}^{n} D_i \to \overline{X}$ relative to S with D_i/S smooth relative divisors for all i.)

From lemma 4.3.8 above, [SS16] show the following result by base change with a finite étale map.

Proposition 4.3.9 ([SS16] 4.4)**.** *Let k be a finitely generated field extension of $\mathbb{Q}$. Let $j : Y \to W$ be a smooth locally closed subscheme where $W = C_1 \times ... \times C_n$ is a product of hyperbolic curves over k. Let $f : X \to W$ be a k-morphism where X is a smooth variety over k. If there is a (homotopy category) isomorphism $\gamma : X_{\text{et}} \to Y_{\text{et}}$ and a commutative diagram*

$$\begin{tikzcd} X_{\text{et}} \arrow[d, "\gamma"] \arrow[dr, "f_{\text{et}}"] & \\ Y_{\text{et}} \arrow[r, "j_{\text{et}}"] & W_{\text{et}} \end{tikzcd}$$

in $\mathrm{Ho}(\text{pro-}\mathbf{ss})$*, then f factors through j:*

$$\begin{tikzcd} X \arrow[d, dashed, "\exists g"'] \arrow[dr, "f"] & \\ Y \arrow[r, "j"] & W. \end{tikzcd}$$

4.3.b. The main theorem

Theorem 4.3.10 ([SS16] 4.7)**.** *Let k be a finitely generated extension of $\mathbb{Q}$. Let X, Y (and Z) be smooth geometrically connected varieties over k that can be embedded as locally closed subschemes into a product of hyperbolic curves over k. Then the natural map $(-)_{\text{et}} : \mathrm{Isom}_k(X, Y) \to \mathrm{Isom}_{\mathrm{Ho}(\text{pro-}\mathbf{ss})/k_{\text{et}}}(X_{\text{et}}, Y_{\text{et}})$ admits a unique retraction r which is functorial. It satisfies the following properties:*

- *(Retraction.) For all k-isomorphisms $g : X \to Y$ we have $r(g_{\text{et}}) = g$.*
- *(Functoriality.) If we have homotopy isomorphisms $\gamma_1 : X_{\text{et}} \to Y_{\text{et}}$ and $\gamma_2 : Y_{\text{et}} \to Z_{\text{et}}$ over k_{et} then $r(\gamma_2)r(\gamma_1) = r(\gamma_2\gamma_1)$.*
- *(Hyperbolic curves.) If γ is a homotopy isomorphism $X \to Y$ over k_{et} and $h : Y \to C$ is a dominant k-morphism to a hyperbolic curve C then $h_{\text{et}}r(\gamma)_{\text{et}} = h_{\text{et}}\gamma$ in* $\mathrm{Ho}(\text{pro-}\mathbf{ss})/k_{\text{et}}$.

Proof. We divide the proof into five parts.

1. (Constructing the map.) Let $j : Y \to W = C_1 \times ... \times C_n$ and $\gamma : X_{\text{et}} \to Y_{\text{et}}$ be as in the assumptions with $p_i : W \to C_i$ the projection maps.

By 4.3.7 there is a unique k-map $f : X \to W$ with $f_{\text{et}} = j_{\text{et}}\gamma$. By 4.3.9 f factors through a map $g : X \to Y$ as $f = jg$. Let $r(\gamma) = g$. It is clear that $(jr(\gamma))_{\text{et}} = f_{\text{et}}$ and thus also equal to $j_{\text{et}}\gamma$. Furthermore $(p_ijr(\gamma))_{\text{et}} = (p_ij)_{\text{et}}\gamma$. Since p_ij is dominant by assumption and γ is an isomorphism, $p_ijr(\gamma)$ is dominant and $(p_ijr(\gamma))_{\text{et}}$ is π_1-open for all i.

2. (The retraction property.) Assume that $\gamma = g_{\text{et}}$ for a k-isomorphism $g : X \to Y$. Then the unique map $X \to W$ factoring through $g : X \to Y$ is precisely $f = jg$ and since it is unique we must have $r(g_{\text{et}}) = g$.

Assume that we have another immersion $j' : Y \to W'$ which is a product of hyperbolic curves C'_j. The same construction gives us a map $f' : X \to W'$ and a map $g' : X \to Y$. By considering the product $W \times W'$, which is also a product of hyperbolic curves, we can apply the construction to the product map (j, j') and obtain a map (f, f') and a factorisation $h : X \to Y$. By projecting onto the two different components we can deduce that $g = h = g'$ is indeed independent of the choice of j.

3. (Functoriality.) Let X, Y, Z and the maps γ_1, γ_2 be as in the assumptions and choose a factor-dominant embedding $j_Z : Z \to V$ into a product V of hyperbolic curves D_i. The maps $p_i j_Z r(\gamma_2\gamma_1) : X \to D_i$ are dominant for all i by assumption. For functoriality we would like to prove that

$$j_Z r(\gamma_2\gamma_1) = j_Z r(\gamma_2) r(\gamma_1)$$

and by applying the homotopy-theoretic version of Mochizuki's theorem, 4.3.5, this corresponds to an equality

$$(j_Z r(\gamma_2\gamma_1))_{\mathrm{et}} = (j_Z r(\gamma_2) r(\gamma_1))_{\mathrm{et}}.$$

Given a factor-dominant $j_Y : Y \to W$ we construct a factor-dominant immersion

$$j = (j_Y, j_Z r(\gamma_2)) : Y \to W \times V.$$

If we denote the second projection by $p : W \times V \to V$ we can use the corresponding versions of the equality $(jr(\gamma))_{\mathrm{et}} = j_{\mathrm{et}}\gamma$ above along with the equality $pj = j_Z r(\gamma_2)$ to write out

$$\begin{aligned}
(j_Z r(\gamma_2\gamma_1))_{\mathrm{et}} &= (j_Z)_{\mathrm{et}}\gamma_2\gamma_1 \\
&= (j_Z r(\gamma_2))_{\mathrm{et}}\gamma_1 \\
&= (pj)_{\mathrm{et}}\gamma_1 \\
&= p_{\mathrm{et}} j_{\mathrm{et}}\gamma_1 \\
&= p_{\mathrm{et}}(jr(\gamma_1))_{\mathrm{et}} \\
&= (pjr(\gamma_1))_{\mathrm{et}} \\
&= (j_Z r(\gamma_2) r(\gamma_1))_{\mathrm{et}}
\end{aligned}$$

which shows functoriality.

Note that the established properties show that $r(\gamma)$ is an isomorphism since the inverse γ^{-1} gives a map $r(\gamma^{-1})$ which is the inverse of $r(\gamma)$.

4. (Hyperbolic curves.) Let $C_1 = C$ and let $C_2, ..., C_n$ be hyperbolic curves with a factor-dominant immersion $j : Y \to W = C_1 \times ... \times C_n$. Let $p : W \to C_1$ be the projection so that $h = pj$ is the map in the assumptions. Then as before we have the equality

$$j_{\mathrm{et}} r(\gamma)_{\mathrm{et}} = j_{\mathrm{et}}\gamma$$

which we can compse with p_{et} to obtain

$$p_{\mathrm{et}} j_{\mathrm{et}} r(\gamma)_{\mathrm{et}} = p_{\mathrm{et}} j_{\mathrm{et}}\gamma$$

and thus by $p_{\mathrm{et}} j_{\mathrm{et}} = h_{\mathrm{et}}$ we have

$$h_{\mathrm{et}} r(\gamma)_{\mathrm{et}} = h_{\mathrm{et}}\gamma$$

which is the desired property.

5. (Uniqueness of r.) For $j : Y \to W$ as above and $f_i : Y \to C_i$ the composition with the ith projection,

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[dr, "f_i"'] & W \arrow[d] \\ & C_i \end{tikzcd}$$

we have for a given $\gamma : X_{\text{et}} \to Y_{\text{et}}$ that

$$(f_i r(\gamma))_{\text{et}} = (f_i)_{\text{et}} r(\gamma)_{\text{et}} = (f_i)_{\text{et}} \gamma$$

by the hyperbolic curve property above. Using 4.3.5 with $h = f_i$ as the dominant morphism, we see that $(f_i)_{\text{et}} r(\gamma)_{\text{et}} = (f_i)_{\text{et}} \gamma$. This uniquely determines a map $g : X \to Y$ such that $(jg)_{\text{et}} = \gamma$ and thus the retraction r is unique. □

As a beautiful direct corollary we can now state the following "weakly" anabelian result:

Corollary 4.3.11 ([SS16] 1.3)**.** *Let k, X, Y be as above. If $X_{\text{et}} \cong Y_{\text{et}}$ in $\text{Ho}(\text{pro-}\mathbf{ss})/k_{\text{et}}$, then $X \cong Y$ as k-varieties.*

4.3.c. The rest of the paper

For X, Y over k consider as before the map

$$\phi_{Y,X} : \text{Isom}_k(Y, X) \to \text{Isom}_{\text{Ho}(\text{pro-}\mathbf{ss}_*)/k_{\text{et}}}(Y_{\text{et}}, X_{\text{et}}).$$

Thus far, we have seen that

- If X, Y are hyperbolic curves the map $\phi_{Y,X}$ is a bijection (4.3.5). This holds for k sub-p-adic.
- If X, Y are geometrically connected over k and embeddable into a product of hyperbolic curves, then $\phi_{Y,X}$ is a split injection (4.3.10). This holds for $k| \mathbb{Q}$ finitely generated.
- With the assumptions of the previous point, if $X_{\text{et}} \cong_{k_{\text{et}}} Y_{\text{et}}$ then $X \cong_k Y$ (4.3.11).
- If X is of type $K(\pi, 1)$ this implies the first conjecture of 4.1.1 with the "weak" condition.

Although we will not go through it in any detail here, Stix and Schmidt proceed to prove several other results in their paper. After proving the theorem 4.3.10 in their fourth section, they devote the fifth section to some further functoriality properties of the retraction. In the sixth section, they turn to *strongly hyperbolic Artin neighbourhoods*.

Definition 4.3.12. A *strongly hyperbolic Artin neighbourhood* is a smooth variety X over k such that there exists maps

$$X = X_n \to X_{n-1} \to ... \to X_0 = \text{Spec}\, k$$

fulfilling two conditions:

- For all i the map $X_i \to X_{i-1}$ is an elementary fibration into a hyperbolic curve.
- X_i has an immersion into a product of hyperbolic curves for all i.

The discussion about strongly hyperbolic Artin-neighbourhoods culminates in the following theorem.

Theorem 4.3.13 ([SS16] 1.5)**.** *Let X, Y be a strongly hyperbolic Artin-neighbourhood over k. Then $\phi_{Y,X}$ is a bijection.*

This implies the second conjecture of 4.1.1 with the "strong" condition. For the full statement, see Corollary 1.6 of [SS16], which is proven in section 6 after 6.2. In their seventh and final section, they prove a birational absolute version of the main theorem:

Theorem 4.3.14 ([SS16] 7.2)**.** *With k a finitely generated extension of $\mathbb{Q}$ and X, Y smooth varieties over k whose every connected component can be embedded into a product of hyperbolic curves over their fields of constants, the map*

$$\text{Isom}_k(X, Y) \to \text{Isom}_{\text{Ho}(\text{pro-}\mathbf{ss})/k_{\text{et}}}(X_{\text{et}}, Y_{\text{et}})$$

is a split injection with a functorial retraction, and if the connected components are strongly hyperbolic Artin neighbourhoods it is a bijection.

4.4. Coda

Remark 4.4.1. (*Connections to IUT and the abc-conjecture*) Mochizuki is probably most famous for his work on Inter-universal Teichmüller theory, which he applies to the abc-conjecture, and this work is not unrelated to the material in the current thesis. Let us present a quick overview of the situation.

There are three main branches of prerequisites for the four IUT papers. The first one is the topic of "semi-graphs of anabelioids". Here, a *connected anabelioid* is a category of G-sets for a profinite group G, which is what topos theorists would call a *classifying topos* for the group. Mochizuki's naming derives from the fact that these categories as toposes are fully determined by G, similar to anabelian curves being fully determined by their fundamental group.

Secondly, he introduces objects called *frobenioids*, which loosely can be described as an abstraction of divisors on a scheme, and is part of his dichotomy between "étale-like" and "frobenius-like" structures. This is related to monoid theory and to log-schemes, which he uses to prove the main theorem of [Moc96]. As the third and last branch, he uses concepts from what he calls "absolute anabelian geometry" which is an extension of Theorem 4.1.4.

With these concepts and methods that rely heavily on various properties of topological groups, he formalises and approaches the concept of reconstructibility with novel tools such as "Hodge theaters". These concepts are then applied in ways that could prove to have profound consequences not only for anabelian considerations, but for all of algebraic geometry.

Remark 4.4.2. Let X be a connected smooth projective variety over k with rational points $X(k)$. Functoriality gives a map

$$X(k) \to \hom(k_{\text{et}}, X_{\text{et}})$$

by taking rational points $k \to X$ to maps $k_{\text{et}} \to X_{\text{et}}$ in a suitable homotopy category of profinite objects over k_{et}. We can interpret k_{et} as the classifying space BG for $G = \text{Gal}(k)$. Thus there is a map $X(k) \to \hom(BG, X_{\text{et}})$. This can be interpreted as

$$X(k) \to \pi_0((X_{\text{et}})^{hG})$$

where the right hand side is the connected components of the continuous homotopy fixed points of X_{et}. Harpaz and Schlank have applied this to obstructions: the idea is that if the right hand side is empty, which is a homotopy theoretic question, then the left hand side $X(k)$ has to be empty as well. We can also raise the following general question: Is the map $X(k) \to \pi_0((X_{\text{et}})^{hG})$ surjective?

For connected smooth projective curves of genus > 1 this is a variant of the section conjecture. The hope is that this line of thought will give new insight with tools of homotopy theory traditionally unavailable in algebraic geometry. An example of this is the paper by Stix and Schmidt [SS16] that we have surveyed in this thesis. A very nice overview of these matters can be found in chapter 2 of Stix' book on the section conjecture, [Sti13].

Remark 4.4.3. Schmidt shows in [Sch12] that under certain conditions, the étale homotopy type functor factors through the Morel-Voevodsky $\mathbb{A}^1$-homotopy type. In particular, this means that it would be impossible to extract more information from the étale homotopy type of a scheme than what is contained in the $\mathbb{A}^1$-homotopy type. Furthermore, anabelian schemes would have to be $\mathbb{A}^1$-local, which indeed is the case for hyperbolic curves. Hence, it could be seen as more natural to ask the more general "motivic anabelian" question: which schemes are reconstructible from the $\mathbb{A}^1$-homotopy type? ○

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